Diagrammatic notations for interactive theorem proving

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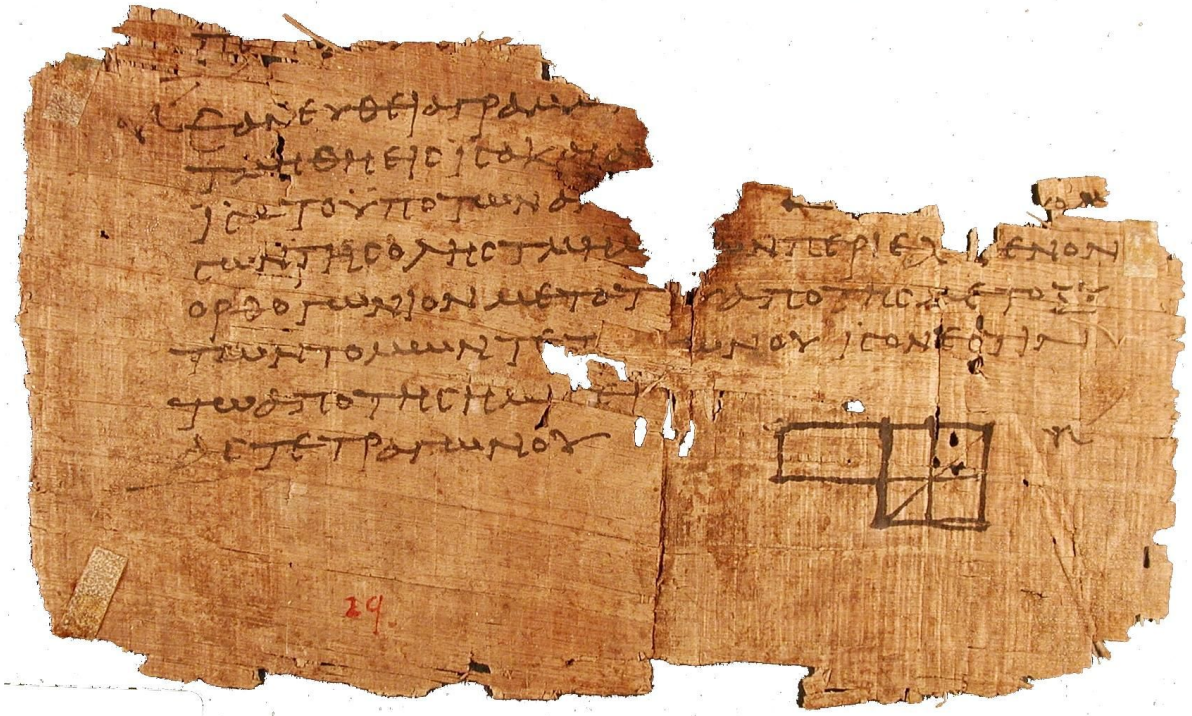
Proofs through the ages

Proofs through the ages



Euclid's proof of

$$ab + \left(\frac{a-b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2$$

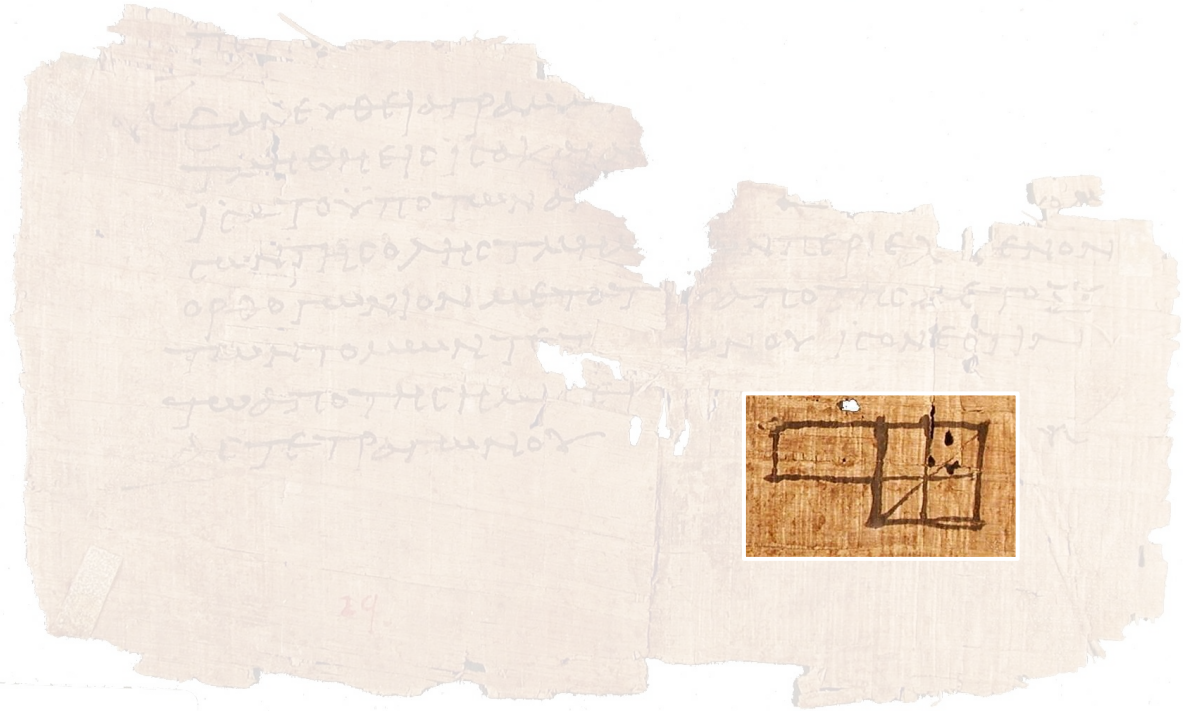
Elements, Euclid c. 300 BCE

scribbled on papyrus by unknown author, c. 100 CE

discovered by Grenfell and Hunt, 1896

photographed by Casselman, 1994

Proofs through the ages



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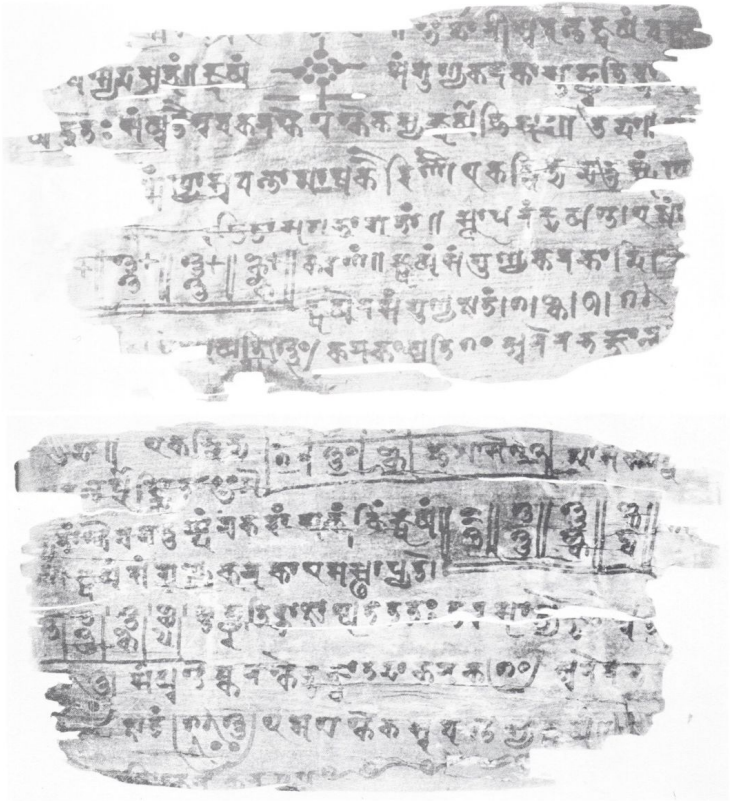

solving cubic equations (?)

Bakhshali manuscript, c. 300–900 CE (disputed)
discovered by anonymous peasant, 1881
photographed by Kaye, 1927
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Proofs through the ages

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Proofs through the ages

solving cubic equations (?)

… and one of the oldest physical records of the numeral zero

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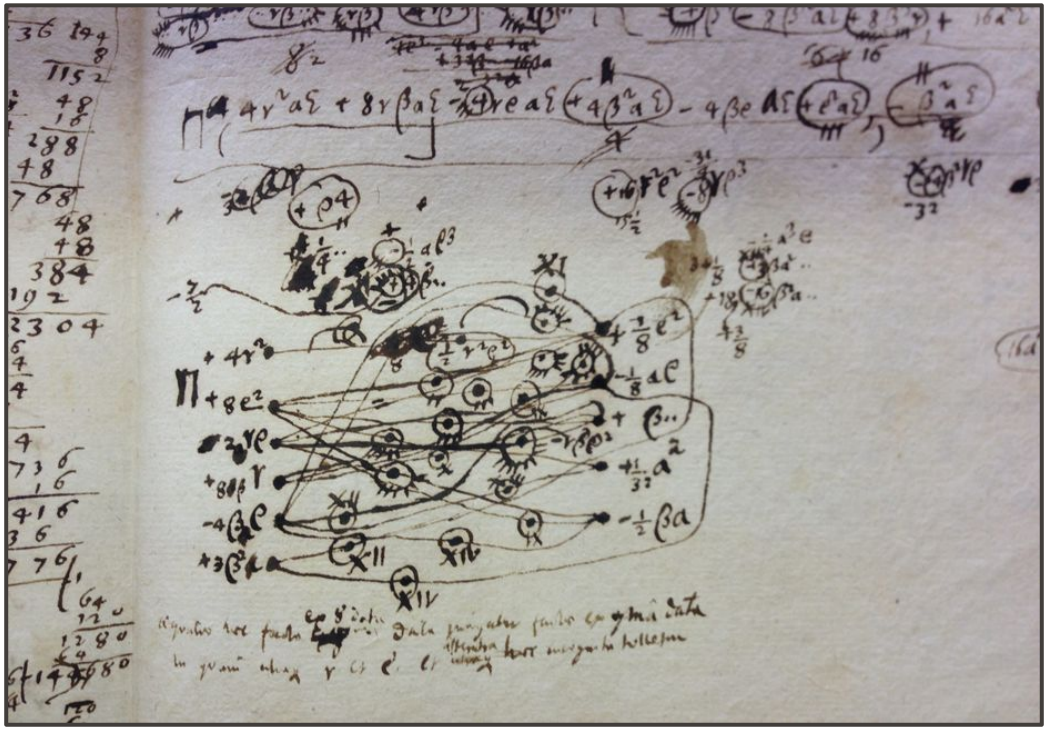
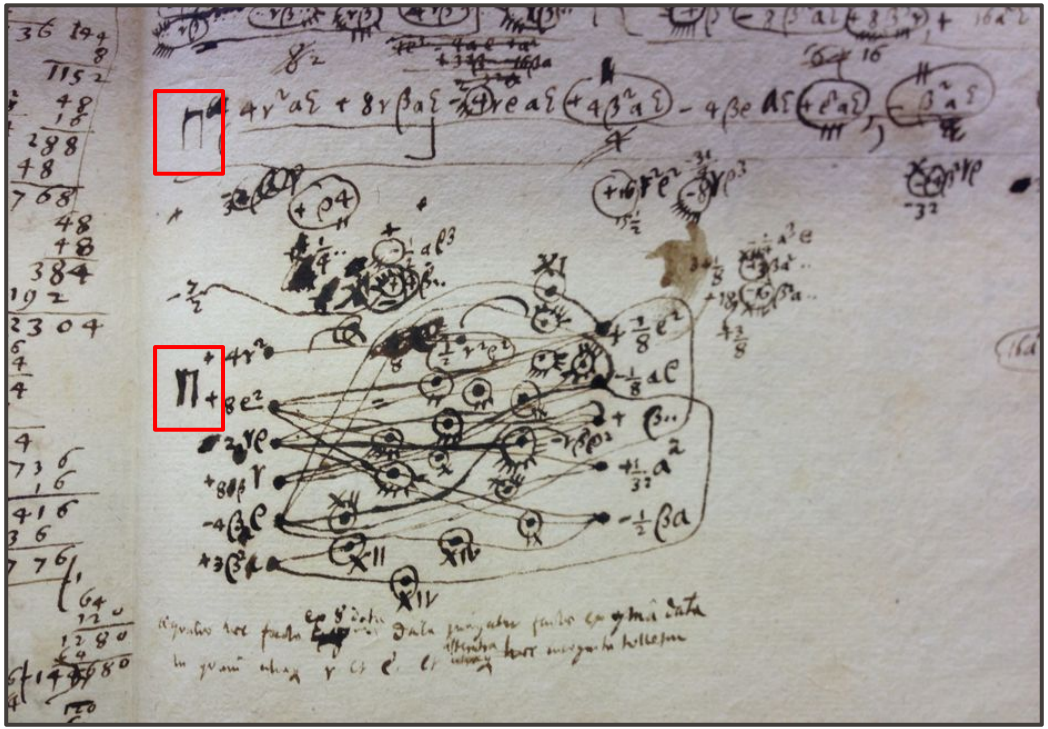
Proofs through the ages



Leibniz's diagrammatic calculus for… not sure exactly what

personal notes, Leibniz, c. 1675
photographed by Wolfram, 2013

Proofs through the ages



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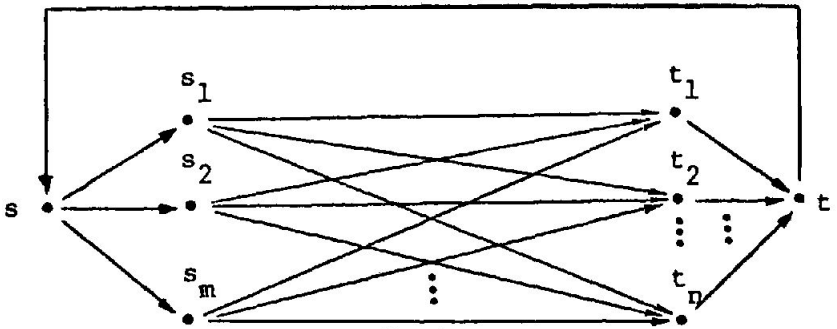


FIG. 1

THEOREM 8. *The flow f is extreme among maximum flows for the network of Figure 1, if and only if there exist* $u_0, u_1, \cdots, u_m$ *and* $v_0, v_1, \cdots, v_m$ *such that*

$$u_i - v_j + d_{ij} \geq 0,\ i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n, \tag{5a}$$

$$u_i - v_j + d_{ij} > 0 \Rightarrow f_{ij} = 0,\ i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n, \tag{5b}$$

$$u_0 > u_i \Rightarrow f_{0i} = 0, \tag{5c}$$

$$u_0 < u_i \Rightarrow f_{0i} = a_i, \tag{5d}$$

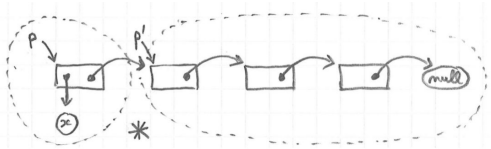
$$v_j > v_0 \Rightarrow f_{j0} = 0, \tag{5e}$$

$$v_j < v_0 \Rightarrow f_{j0} = b_j \tag{5f}$$

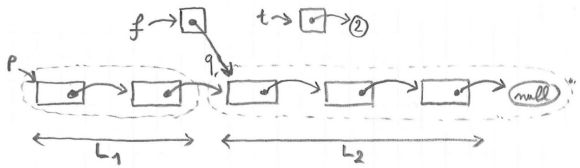
Edmonds-Karp algorithm
for max flow in a network

Theoretical Improvements in Algorithmic Efficiency for Network Flow Problems, Edmonds and Karp, 1972

Proofs through the ages



$$p \rightsquigarrow \mathsf{Mlist}\, L \;\equiv\; \mathsf{match}\, L\, \mathsf{with}$$
$$\mid \mathsf{nil} \;\Rightarrow\; [p = \mathsf{null}]$$
$$\mid x :: L' \;\Rightarrow\; \exists p'.\quad p \rightsquigarrow \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \star p' \rightsquigarrow \mathsf{Mlist}\, L'$$



Loop invariant:

$$\exists L_1 L_2 q.\quad [L = L_1 +\!\!+ L_2] \star (t \mapsto \mathsf{length}\, L_1) \star (f \mapsto q) \star (p \rightsquigarrow \mathsf{MlistSeg}\, q\, L_1) \star (q \rightsquigarrow \mathsf{Mlist}\, L_2)$$

proof of `length` of a linked list with separation logic

Separation Logic, Charguéraud, 2016

Proofs through the ages

It’s 2023!

Proofs through the ages

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Proof assistants are becoming popular with mathematicians and computer scientists.

Proofs through the ages

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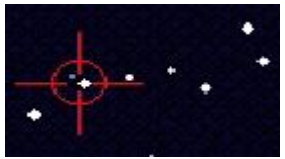
HOL
Interactive Theorem Prover



LEAN

Agda

Proofs through the ages

```
Lemma prop_14 : forall A B C D P Q R S T U,
    TS A B C D ->
    Per P Q R ->
    SumA A B C A B D S T U ->
    SumA P Q R P Q R S T U ->
    Bet C B D.
Proof.
  intros A B C D P Q R S T U HTS HP HSuma1 HSuma2.
  apply (bet_conga__bet S T U).
    apply (per2_suma__bet P Q R P Q R); assumption.
  apply (suma2__conga A B C A B D).
    assumption.
  apply suma_left_comm. apply ts__suma. apply HTS.
Qed.
```

```
1 goal (ID 248)

- Tn : Tarski_neutral_dimensionless
- TnEQD : Tarski_neutral_dimensionless_with_decidable_point_equality Tn
- A, B, C, D, P, Q, R, S, T, U : Tpoint
- HTS : TS A B C D
- HP : Per P Q R
- HSuma1 : SumA A B C A B D S T U
- HSuma2 : SumA P Q R P Q R S T U
  ============================
  Bet C B D
```

GeoCoq, Narboux et al.

Proofs through the ages

```
Lemma prop_14 : forall A B C D P Q R S T U,
    TS A B C D ->
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    Bet C B D.
Proof.
  intros A B C D P Q R S T U HTS HP HSuma1 HSuma2.
  apply (bet_conga__bet S T U).
    apply (per2_suma__bet P Q R P Q R); assumption.
  apply (suma2__conga A B C A B D).
    assumption.
  apply suma_left_comm. apply ts__suma. apply HTS.
Qed.
```

```
2 goals (ID 251)

- Tn : Tarski_neutral_dimensionless
- TnEQD : Tarski_neutral_dimensionless_with_decidable_point_equality Tn
- A, B, C, D, P, Q, R, S, T, U : Tpoint
- HTS : TS A B C D
- HP : Per P Q R
- HSuma1 : SumA A B C A B D S T U
- HSuma2 : SumA P Q R P Q R S T U
  ============================
  Bet S T U

goal 2 (ID 252) is:
 CongA S T U C B D
```

GeoCoq, Narboux et al.

Proofs through the ages

```
Lemma prop_14 : forall A B C D P Q R S T U,
    TS A B C D ->
    Per P Q R ->
    SumA A B C A B D S T U ->
    SumA P Q R P Q R S T U ->
    Bet C B D.
Proof.
  intros A B C D P Q R S T U HTS HP HSuma1 HSuma2.
  apply (bet_conga__bet S T U).
    apply (per2_suma__bet P Q R P Q R); assumption.
  apply (suma2__conga A B C A B D).
    assumption.
  apply suma_left_comm. apply ts__suma. apply HTS.
Qed.
```

```
1 goal (ID 252)

- Tn : Tarski_neutral_dimensionless
- TnEQD : Tarski_neutral_dimensionless_with_decidable_point_equality Tn
- A, B, C, D, P, Q, R, S, T, U : Tpoint
- HTS : TS A B C D
- HP : Per P Q R
- HSuma1 : SumA A B C A B D S T U
- HSuma2 : SumA P Q R P Q R S T U
  ============================
  CongA S T U C B D
```

GeoCoq, Narboux et al.

Proofs through the ages

```
Lemma prop_14 : forall A B C D P Q R S T U,
    TS A B C D ->
    Per P Q R ->
    SumA A B C A B D S T U ->
    SumA P Q R P Q R S T U ->
    Bet C B D.
Proof.
  intros A B C D P Q R S T U HTS HP HSuma1 HSuma2.
  apply (bet_conga__bet S T U).
    apply (per2_suma__bet P Q R P Q R); assumption.
  apply (suma2__conga A B C A B D).
    assumption.
  apply suma_left_comm. apply ts__suma. apply HTS.
Qed.
```

```
2 goals (ID 260)

- Tn : Tarski_neutral_dimensionless
- TnEQD : Tarski_neutral_dimensionless_with_decidable_point_equality Tn
- A, B, C, D, P, Q, R, S, T, U : Tpoint
- HTS : TS A B C D
- HP : Per P Q R
- HSuma1 : SumA A B C A B D S T U
- HSuma2 : SumA P Q R P Q R S T U
  ============================
  SumA A B C A B D S T U

goal 2 (ID 261) is:
 SumA A B C A B D C B D
```

GeoCoq, Narboux et al.

Proofs through the ages

```
Lemma prop_14 : forall A B C D P Q R S T U,
    TS A B C D ->
    Per P Q R ->
    SumA A B C A B D S T U ->
    SumA P Q R P Q R S T U ->
    Bet C B D.
Proof.
  intros A B C D P Q R S T U HTS HP HSuma1 HSuma2.
  apply (bet_conga__bet S T U).
    apply (per2_suma__bet P Q R P Q R); assumption.
  apply (suma2__conga A B C A B D).
    assumption.
 apply suma_left_comm. apply ts__suma. apply HTS.
Qed.
```

```
1 goal (ID 261)

- Tn : Tarski_neutral_dimensionless
- TnEQD : Tarski_neutral_dimensionless_with_decidable_point_equality Tn
- A, B, C, D, P, Q, R, S, T, U : Tpoint
- HTS : TS A B C D
- HP : Per P Q R
- HSuma1 : SumA A B C A B D S T U
- HSuma2 : SumA P Q R P Q R S T U
  ============================
  SumA A B C A B D C B D
```

GeoCoq, Narboux et al.

Proofs through the ages

```
Lemma prop_14 : forall A B C D P Q R S T U,
    TS A B C D ->
    Per P Q R ->
    SumA A B C A B D S T U ->
    SumA P Q R P Q R S T U ->
    Bet C B D.
Proof.
  intros A B C D P Q R S T U HTS HP HSuma1 HSuma2.
  apply (bet_conga__bet S T U).
    apply (per2_suma__bet P Q R P Q R); assumption.
  apply (suma2__conga A B C A B D).
    assumption.
  apply suma_left_comm. apply ts__suma. apply HTS.
Qed.
```

```
1 goal (ID 264)

- Tn : Tarski_neutral_dimensionless
- TnEQD : Tarski_neutral_dimensionless_with_decidable_point_equality Tn
- A, B, C, D, P, Q, R, S, T, U : Tpoint
- HTS : TS A B C D
- HP : Per P Q R
- HSuma1 : SumA A B C A B D S T U
- HSuma2 : SumA P Q R P Q R S T U
  ============================
  SumA C B A A B D C B D
```

GeoCoq, Narboux et al.

Proofs through the ages

```
Lemma prop_14 : forall A B C D P Q R S T U,
    TS A B C D ->
    Per P Q R ->
    SumA A B C A B D S T U ->
    SumA P Q R P Q R S T U ->
    Bet C B D.
Proof.
  intros A B C D P Q R S T U HTS HP HSuma1 HSuma2.
  apply (bet_conga__bet S T U).
    apply (per2_suma__bet P Q R P Q R); assumption.
  apply (suma2__conga A B C A B D).
    assumption.
  apply suma_left_comm. apply ts__suma. apply HTS.
Qed.
```

```
1 goal (ID 267)

- Tn : Tarski_neutral_dimensionless
- TnEQD : Tarski_neutral_dimensionless_with_decidable_point_equality Tn
- A, B, C, D, P, Q, R, S, T, U : Tpoint
- HTS : TS A B C D
- HP : Per P Q R
- HSuma1 : SumA A B C A B D S T U
- HSuma2 : SumA P Q R P Q R S T U
  ============================
  TS A B C D
```

GeoCoq, Narboux et al.

Proofs through the ages

```
Lemma concat_spec : forall A `{Enc A}
                           (p1 p2:loc) (L1 L2:list A),
  SPEC (concat p1 p2)
    PRE (p1 ~> Stack L1 \* p2 ~> Stack L2)
    POST (fun (u:unit) => p1 ~> Stack (L1 ++ L2)
                           \* p2 ~> Stack (@nil A)).
Proof using.
  xcf. xunfold Stack. xapp. xapp. xapp.
  rewrite <- (Stack_eq p2). xapp. xsimpl.
Qed.
```

```
1 goal (ID 127)

  ============================
  forall (A : Type) (H : Enc A) p1 p2 (L1 L2 : list A),
  Triple
    (Trm_apps concat
       (cons {| dyn_type := loc; dyn_enc := Enc_loc; dyn_value := p1 |}
          (cons {| dyn_type := loc; dyn_enc := Enc_loc; dyn_value := p2 |}
             nil))) (hstar (repr (Stack L1) p1) (repr (Stack L2) p2))
    (fun _ : unit =>
     hstar (repr (Stack (app L1 L2)) p1) (repr (Stack nil) p2))
```

Proofs through the ages

```
Lemma concat_spec : forall A `{Enc A}
                           (p1 p2:loc) (L1 L2:list A),
  SPEC (concat p1 p2)
    PRE (p1 ~> Stack L1 \* p2 ~> Stack L2)
    POST (fun (u:unit) => p1 ~> Stack (L1 ++ L2)
                           \* p2 ~> Stack (@nil A)).
Proof using.
  xcf. xunfold Stack. xapp. xapp. xapp.
  rewrite <- (Stack_eq p2). xapp. xsimpl.
Qed.
```

```
1 focused goal
(shelved: 2) (ID 154)

- A : Type
- H : Enc A
- p1, p2 : loc
- L1, L2 : list A
  ============================
  himpl (hstar (repr (Stack L1) p1) (repr (Stack L2) p2))
    (Wptag
       (Wpgen_let_trm
          (Wptag
             (Wpgen_app (list ?A_) infix_emark__
                (cons {| dyn_type := ref_ (list ?A_); dyn_enc := Enc_loc;
dyn_value := p2 |} nil)))
          (fun x1__ : list ?A_ =>
           Wptag
             (Wpgen_let_trm
                (Wptag
                   (Wpgen_app (list ?A_) infix_emark__
                      (cons {| dyn_type := ref_ (list ?A_); dyn_enc := Enc
_loc; dyn_value := p1 |} nil)))
                (fun x0__ : list ?A_ =>
                 Wptag
                   (Wpgen_seq
                      (Wptag
                         (Wpgen_app unit infix_colon_eq__
                            (cons {| dyn_type := ref_ (list ?A_); dyn_enc
:= Enc_loc; dyn_value := p1 |}
                               (cons {| dyn_type := list ?A_; dyn_enc := E
nc_list; dyn_value := app x0__ x1__ |} nil))))
                      (Wptag
                         (Wpgen_app unit clear
                            (cons {| dyn_type := ref_ (list ?A_); dyn_enc
:= Enc_loc; dyn_value := p2 |} nil)))))))
          Enc_unit (fun _ : unit => hstar (hstar (repr (Stack (app L1 L2)) p1
) (repr (Stack nil) p2)) hgc))
```

Proofs through the ages

```
Lemma concat_spec : forall A `{Enc A}
                           (p1 p2:loc) (L1 L2:list A),
  SPEC (concat p1 p2)
    PRE (p1 ~> Stack L1 \* p2 ~> Stack L2)
    POST (fun (u:unit) => p1 ~> Stack (L1 ++ L2)
                           \* p2 ~> Stack (@nil A)).
Proof using.
  xcf. xunfold Stack. xapp. xapp. xapp.
  rewrite <- (Stack_eq p2). xapp. xsimpl.
Qed.
```

```
1 focused goal
(shelved: 2) (ID 168)

- A : Type
- H : Enc A
- p1, p2 : loc
- L1, L2 : list A
  ============================
  himpl (hstar (repr (Ref L1) p1) (repr (Ref L2) p2))
    (Wptag
       (Wpgen_let_trm
          (Wptag
             (Wpgen_app (list ?A_) infix_emark__
                (cons {| dyn_type := ref_ (list ?A_); dyn_enc := Enc_loc;
*dyn_value := p2 |} nil)))
          (fun x1__ : list ?A_ =>
           Wptag
             (Wpgen_let_trm
                (Wptag
                   (Wpgen_app (list ?A_) infix_emark__
                      (cons {| dyn_type := ref_ (list ?A_); dyn_enc := Enc
*_loc; dyn_value := p1 |} nil)))
                (fun x0__ : list ?A_ =>
                 Wptag
                   (Wpgen_seq
                      (Wptag
                         (Wpgen_app unit infix_colon_eq__
                            (cons {| dyn_type := ref_ (list ?A_); dyn_enc
*:= Enc_loc; dyn_value := p1 |}
                               (cons {| dyn_type := list ?A_; dyn_enc := E
*nc_list; dyn_value := app x0__ x1__ |} nil))))
                      (Wptag
                         (Wpgen_app unit clear
                            (cons {| dyn_type := ref_ (list ?A_); dyn_enc
*:= Enc_loc; dyn_value := p2 |} nil))))))))
       Enc_unit (fun _ : unit => hstar (hstar (repr (Ref (app L1 L2)) p1)
*(repr (Ref nil) p2)) hgc))
```

Proofs through the ages

```
Lemma concat_spec : forall A `{Enc A}
                          (p1 p2:loc) (L1 L2:list A),
  SPEC (concat p1 p2)
    PRE (p1 ~> Stack L1 \* p2 ~> Stack L2)
    POST (fun (u:unit) => p1 ~> Stack (L1 ++ L2)
                           \* p2 ~> Stack (@nil A)).
Proof using.
  xcf. xunfold Stack. xapp. xapp. xapp.
  rewrite <- (Stack_eq p2). xapp. xsimpl.
Qed.
```

```
1 goal (ID 2077)

- A : Type
- H : Enc A
- p1, p2 : loc
- L1, L2 : list A
  ============================
  PRE (p1 ~~> (L1 ++ L2) \* p2 ~~> L2)
  CODE (Wpgen_app unit clear ((Dyn p2) :: nil))
  POST (fun _ : unit => (p1 ~~> (L1 ++ L2) \* p2 ~~> nil) \* \GC)
```

cfml2, Charguéraud

Proofs through the ages

```
Lemma concat_spec : forall A `{Enc A}
                          (p1 p2:loc) (L1 L2:list A),
  SPEC (concat p1 p2)
    PRE (p1 ~> Stack L1 \* p2 ~> Stack L2)
    POST (fun (u:unit) => p1 ~> Stack (L1 ++ L2)
                           \* p2 ~> Stack (@nil A)).
Proof using.
  xcf. xunfold Stack. xapp. xapp. xapp.
  rewrite <- (Stack_eq p2). xapp. xsimpl.
Qed.
```

```
1 goal (ID 2078)

- A : Type
- H : Enc A
- p1, p2 : loc
- L1, L2 : list A
  ============================
  PRE (p1 ~~> (L1 ++ L2) \* p2 ~> Stack L2)
  CODE (Wpgen_app unit clear ((Dyn p2) :: nil))
  POST (fun _ : unit => (p1 ~~> (L1 ++ L2) \* p2 ~~> nil) \* \GC)
```

cfml2, Charguéraud

Proofs through the ages

```
Lemma concat_spec : forall A `{Enc A}
                          (p1 p2:loc) (L1 L2:list A),
  SPEC (concat p1 p2)
    PRE (p1 ~> Stack L1 \* p2 ~> Stack L2)
    POST (fun (u:unit) => p1 ~> Stack (L1 ++ L2)
                           \* p2 ~> Stack (@nil A)).
Proof using.
  xcf. xunfold Stack. xapp. xapp. xapp.
  rewrite <- (Stack_eq p2). xapp. xsimpl.
Qed.
```

```
1 goal (ID 3173)

- A : Type
- H : Enc A
- p1, p2 : loc
- L1, L2 : list A
  ============================
  p2 ~> Stack nil \* p1 ~~> (L1 ++ L2) ==> (p1 ~~> (L1 ++ L2) \* p2 ~~> nil) \* \GC
```

cfml2, Charguéraud

Proofs through the ages

```
Lemma concat_spec : forall A `{Enc A}
                          (p1 p2:loc) (L1 L2:list A),
  SPEC (concat p1 p2)
    PRE (p1 ~> Stack L1 \* p2 ~> Stack L2)
    POST (fun (u:unit) => p1 ~> Stack (L1 ++ L2)
                          \* p2 ~> Stack (@nil A)).
Proof using.
  xcf. xunfold Stack. xapp. xapp. xapp.
  rewrite <- (Stack_eq p2). xapp. xsimpl.
Qed.
```

```
1 goal (ID 3173)

- A : Type
- H : Enc A
- p1, p2 : loc
- L1, L2 : list A
  ============================
  p2 ~> Stack nil \* p1 ~~> (L1 ++ L2) ==> (p1 ~~> (L1 ++ L2) \* p2 ~~> nil) \* \GC
```

$$(f_2 \mapsto \langle d_1, c_2\rangle \quad \star\, p_1 \mapsto \langle f_1, b_2\rangle \quad \star\, p_2 \mapsto \langle f_2, b_2\rangle$$
$$\star\, c_2 \mapsto \langle\langle b_2, L_2'\rangle\rangle \quad \star\, b_2 \mapsto \langle d_2, \mathrm{null}\rangle \quad \star\, f_1 \mapsto \langle\langle b_1, L_1\rangle\rangle)$$

$$\exists L_1 L_2 q. \quad [L = L_1 +\!\!+ L_2] \star (t \mapsto \mathsf{length}\, L_1) \star (f \mapsto q)$$
$$\star\, (p \rightsquigarrow \mathsf{MlistSeg}\, q\, L_1) \star (q \rightsquigarrow \mathsf{Mlist}\, L_2)$$

Proofs through the ages

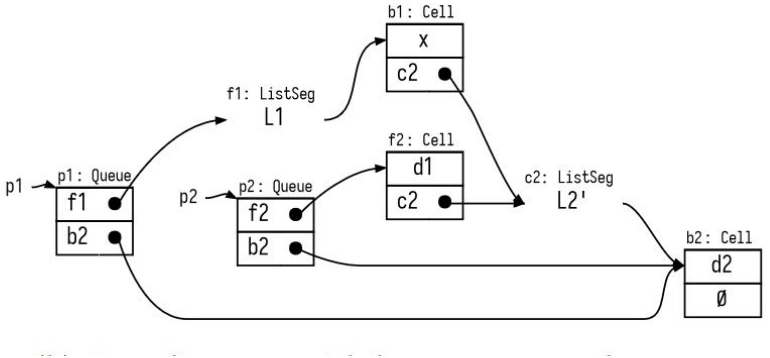
```
Lemma concat_spec : forall A `{Enc A}
                      (p1 p2:loc) (L1 L2:list A),
  SPEC (concat p1 p2)
    PRE (p1 ~> Stack L1 \* p2 ~> Stack L2)
    POST (fun (u:unit) => p1 ~> Stack (L1 ++ L2)
                          \* p2 ~> Stack (@nil A)).
Proof using.
  xcf. xunfold Stack. xapp. xapp. xapp.
  rewrite <- (Stack_eq p2). xapp. xsimpl.
Qed.
```

```
1 goal (ID 3173)

- A : Type
- H : Enc A
- p1, p2 : loc
- L1, L2 : list A
  ============================
  p2 ~> Stack nil \* p1 ~~> (L1 ++ L2) ==> (p1 ~~> (L1 ++ L2) \* p2 ~~> nil) \* \GC
```

$$(f_2 \mapsto \langle d_1, c_2\rangle \quad \star\, p_1 \mapsto \langle f_1, b_2\rangle \quad \star\, p_2 \mapsto \langle f_2, b_2\rangle$$
$$\star\, c_2 \mapsto \langle\langle b_2, L_2'\rangle\rangle \quad \star\, b_2 \mapsto \langle d_2, \mathrm{null}\rangle \quad \star\, f_1 \mapsto \langle\langle b_1, L_1\rangle\rangle)$$

$$\exists L_1 L_2 q. \quad [L = L_1 +\!\!+ L_2] \star (t \mapsto \mathsf{length}\, L_1) \star (f \mapsto q)$$
$$\star\, (p \rightsquigarrow \mathsf{MlistSeg}\, q\, L_1) \star (q \rightsquigarrow \mathsf{Mlist}\, L_2)$$



Plan for this talk

- We need **diagrammatic notations** in computer-assisted proofs, but they are mostly missing.

Plan for this talk

- We need **diagrammatic notations** in computer-assisted proofs, but they are mostly missing. Why?

Plan for this talk

- We need **diagrammatic notations** in computer-assisted proofs, but they are mostly missing. Why?
- Let's learn from a success story: **text notations**.

Plan for this talk

- We need **diagrammatic notations** in computer-assisted proofs, but they are mostly missing. Why?
- Let's learn from a success story: **text notations**.
  → Identify **five principles**.

Plan for this talk

- We need **diagrammatic notations** in computer-assisted proofs, but they are mostly missing. Why?
- Let's learn from a success story: **text notations**.
  → Identify **five principles**.
- What might these look like for diagrams?

```
(sep (MCell f2 d1 c2) (sep (MCell p1
  f1 b2) (sep (MCell p2 f2 b2) (sep
    (MCell b1 x c2) (sep (MListSeg c2 b2
      L2') (sep (MCell b2 d2 null)
        (MListSeg f1 b1 L1)))))))
```

```
    (sep (MCell f2 d1 c2) (sep (MCell p1
      f1 b2) (sep (MCell p2 f2 b2) (sep
        (MCell b1 x c2) (sep (MListSeg c2 b2
          L2') (sep (MCell b2 d2 null)
            (MListSeg f1 b1 L1)))))))

Notation "p1 '\*' p2" := sep p1 p2.
```

```
(sep (MCell f2 d1 c2) (sep (MCell p1
  f1 b2) (sep (MCell p2 f2 b2) (sep
    (MCell b1 x c2) (sep (MListSeg c2 b2
      L2') (sep (MCell b2 d2 null)
        (MListSeg f1 b1 L1)))))))
```

```
Notation "p1 '\*' p2" := sep p1 p2.
```

```
((MCell f2 d1 c2) \* ((MCell p1 f1 b2)
  \* ((MCell p2 f2 b2) \* ((MCell b1 x c2)
  \* ((MListSeg c2 b2 L2') \* ((MCell b2 d2 null)
  \* (MListSeg f1 b1 L1)))))))
```

```
   (sep (MCell f2 d1 c2) (sep (MCell p1
     f1 b2) (sep (MCell p2 f2 b2) (sep
       (MCell b1 x c2) (sep (MListSeg c2 b2
         L2') (sep (MCell b2 d2 null)
           (MListSeg f1 b1 L1)))))))

Notation "p1 '\*' p2" := sep p1 p2.

   ((MCell f2 d1 c2) \* ((MCell p1 f1 b2)
     \* ((MCell p2 f2 b2) \* ((MCell b1 x c2)
     \* ((MListSeg c2 b2 L2') \* ((MCell b2 d2 null)
     \* (MListSeg f1 b1 L1)))))))

Notation "p '~>'  '<' hd  ','  tl  '>'"  := MCell p hd tl.
Notation "p '~>' '<<' pre ',' rest '>>'" := MListSeg p pre rest.
```

```
(sep (MCell f2 d1 c2) (sep (MCell p1
  f1 b2) (sep (MCell p2 f2 b2) (sep
    (MCell b1 x c2) (sep (MListSeg c2 b2
      L2') (sep (MCell b2 d2 null)
        (MListSeg f1 b1 L1)))))))

Notation "p1 '\*' p2" := sep p1 p2.

((MCell f2 d1 c2) \* ((MCell p1 f1 b2)
  \* ((MCell p2 f2 b2) \* ((MCell b1 x c2)
  \* ((MListSeg c2 b2 L2') \* ((MCell b2 d2 null)
  \* (MListSeg f1 b1 L1)))))))

Notation "p '~>'  '<' hd  ','  tl  '>'"  := MCell p hd tl.
Notation "p '~>' '<<' pre ',' rest '>>'" := MListSeg p pre rest.

((f2 ~> <d1, c2>) \* ((p1 ~> <f1, b2>)
  \* ((p2 ~> <f2, b2>) \* ((b1 ~> <x, c2>)
  \* ((c2 ~> <<b2, L2'>>) \* ((b2 ~> <d2, null>)
  \* (f1 ~> <<b1, L1>>)))))))
```

```
  (sep (MCell f2 d1 c2) (sep (MCell p1
    f1 b2) (sep (MCell p2 f2 b2) (sep
      (MCell b1 x c2) (sep (MListSeg c2 b2
        L2') (sep (MCell b2 d2 null)
          (MListSeg f1 b1 L1)))))))

Notation "p1 '\*' p2" := sep p1 p2.

  ((M
```

Low cost, high reward

```
Notation "p '~>'  '<' hd  ','  tl  '>'"  := MCell p hd tl.
Notation "p '~>' '<<' pre ',' rest '>>'" := MListSeg p pre rest.

  ((f2 ~> <d1, c2>) \* ((p1 ~> <f1, b2>)
    \* ((p2 ~> <f2, b2>) \* ((b1 ~> <x, c2>)
    \* ((c2 ~> <<b2, L2'>>) \* ((b2 ~> <d2, null>)
    \* (f1 ~> <<b1, L1>>)))))))
```

```
Notation "p1 '\*' p2" := sep p1 p2.
Notation "p '~>'  '<' hd  ','  tl  '>'"  := MCell p hd tl.
Notation "p '~>' '<<' pre ',' rest '>>'" := MListSeg p pre rest.
```

```
(sep (MCell f2 d1 c2) (sep (MCell p1
  f1 b2) (sep (MCell p2 f2 b2) (sep
    (MCell b1 x c2) (sep (MListSeg c2 b2
      L2') (sep (MCell b2 d2 null)
        (MListSeg f1 b1 L1)))))))
```

```
((f2 ~> <d1, c2>) \* ((p1 ~> <f1, b2>)
  \* ((p2 ~> <f2, b2>) \* ((b1 ~> <x, c2>)
  \* ((c2 ~> <<b2, L2'>>) \* ((b2 ~> <d2, null>)
  \* (f1 ~> <<b1, L1>>)))))))
```

```
Notation "p1 '\*' p2" := sep p1 p2.
Notation "p '~>'  '<' hd  ','  tl  '>'"  := MCell p hd tl.
Notation "p '~>' '<<' pre ',' rest '>>'" := MListSeg p pre rest.
```

```
(sep (MCell p1 h1 t1) (MCell p2 h2 t2))
```

```
((p1 ~> <h1, t1>) \* (p2 ~> <h2, t2>))
```

```
Notation "p1 '\*' p2" := sep p1 p2.
Notation "p '~>'  '<' hd  ','  tl  '>'"  := MCell p hd tl.
Notation "p '~>' '<<' pre ',' rest '>>'" := MListSeg p pre rest.
```

```
(sep (MCell p1 h1 t1) (MCell p2 h2 Lprimes))
```

```
((p1 ~> <h1, t1>) \* (p2 ~> <h2, [2, 3, 5, 7]>))
```

```
Notation "p1 '\*' p2" := sep p1 p2.
Notation "p '~>'  '<' hd  ','  tl  '>'"  := MCell p hd tl.
Notation "p '~>' '<<' pre ',' rest '>>'" := MListSeg p pre rest.
```

```
(sep (MCell p1 h1 t1) (MCell p2 h2 Lprimes))
```

Structure-driven composition

```
((p1 ~> <h1, t1>) \* (p2 ~> <h2, [2, 3, 5, 7]>))
```

```
Notation "p1 '\*' p2" := sep p1 p2.

Notation "p '~>'  '<' hd  ','  tl  '>'"  := MCell p hd tl.

Notation "p '~>' '<<' pre ',' rest '>>'" := MListSeg p pre rest.
```

```
((f2 ~> <d1, c2>) \* ((p1 ~> <f1, b2>)
  \* ((p2 ~> <f2, b2>) \* ((b1 ~> <x, c2>)
  \* ((c2 ~> <<b2, L2'>>) \* ((b2 ~> <d2, null>)
  \* (f1 ~> <<b1, L1>>)))))))
```

```
Notation "p1 '\*' p2" := sep p1 p2
  (right associativity).

Notation "p '~>'  '<' hd  ','  tl  '>'"  := MCell p hd tl.

Notation "p '~>' '<<' pre ',' rest '>>'" := MListSeg p pre rest.
```

```
((f2 ~> <d1, c2>) \* ((p1 ~> <f1, b2>)
  \* ((p2 ~> <f2, b2>) \* ((b1 ~> <x, c2>)
  \* ((c2 ~> <<b2, L2'>>) \* ((b2 ~> <d2, null>)
  \* (f1 ~> <<b1, L1>>)))))))
```

```
Notation "p1 '\*' p2" := sep p1 p2
  (right associativity).

Notation "p '~>'  '<' hd  ','  tl  '>'"  := MCell p hd tl.

Notation "p '~>' '<<' pre ',' rest '>>'" := MListSeg p pre rest.
```

```
(f2 ~> <d1, c2>) \* (p1 ~> <f1, b2>)
  \* (p2 ~> <f2, b2>) \* (b1 ~> <x, c2>)
  \* (c2 ~> <<b2, L2'>>) \* (b2 ~> <d2, null>)
  \* (f1 ~> <<b1, L1>>)
```

```
Notation "p1 '\*' p2" := sep p1 p2
  (right associativity, at level 75).

Notation "p '~>'  '<' hd  ','  tl  '>'"  := MCell p hd tl
  (at level 70).

Notation "p '~>' '<<' pre ',' rest '>>'" := MListSeg p pre rest
  (at level 70).
```

```
(f2 ~> <d1, c2>) \* (p1 ~> <f1, b2>)
  \* (p2 ~> <f2, b2>) \* (b1 ~> <x, c2>)
  \* (c2 ~> <<b2, L2'>>) \* (b2 ~> <d2, null>)
  \* (f1 ~> <<b1, L1>>)
```

```
Notation "p1 '\*' p2" := sep p1 p2
  (right associativity, at level 75).

Notation "p '~>'  '<' hd  ','  tl  '>'"  := MCell p hd tl
  (at level 70).

Notation "p '~>' '<<' pre ',' rest '>>'" := MListSeg p pre rest
  (at level 70).
```

```
f2 ~> <d1, c2> \* p1 ~> <f1, b2>
  \* p2 ~> <f2, b2> \* b1 ~> <x, c2>
  \* c2 ~> <<b2, L2'>> \* b2 ~> <d2, null>
  \* f1 ~> <<b1, L1>>
```

```
Infix "\*" := sep
  (right associativity, at level 75).

Notation "p '~>'  '<' hd  ','  tl  '>'"  := MCell p hd tl
  (at level 70).

Notation "p '~>' '<<' pre ',' rest '>>'" := MListSeg p pre rest
  (at level 70).
```

```
f2 ~> <d1, c2> \* p1 ~> <f1, b2>
  \* p2 ~> <f2, b2> \* b1 ~> <x, c2>
  \* c2 ~> <<b2, L2'>> \* b2 ~> <d2, null>
  \* f1 ~> <<b1, L1>>
```

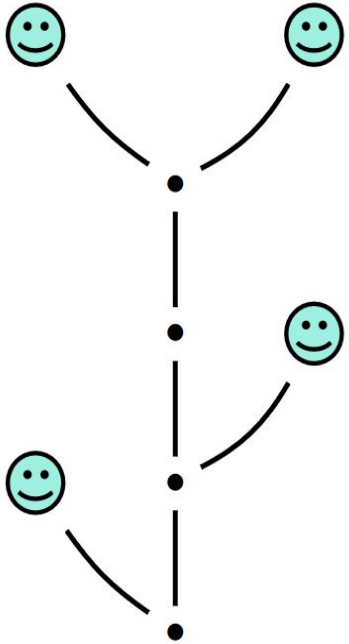
```
Infix "\*" := sep
  (right associativity, at level 75).

Notation "p '~>'  '<' hd  ','  tl  '>'"  := MCell p hd tl
  (at level 70).

Notation "p '~>' '<<' pre ',' rest '>>'" := MListSeg p pre rest
  (at level
```

Good-looking idioms

```
f2 ~> <d1, c2> \* p1 ~> <f1, b2>
  \* p2 ~> <f2, b2> \* b1 ~> <x, c2>
  \* c2 ~> <<b2, L2'>> \* b2 ~> <d2, null>
  \* f1 ~> <<b1, L1>>
```



a hydra

```
(hyd2
  head
  (hyd2
    (hyd1
      (hyd2 head head))
    head))
```

```
(hyd2
  head
  (hyd2
    (hyd1
      (hyd2 head head))
    head))
```

```
(hyd2
  head
  (hyd2
    (hyd1
      (hyd2 head head))
    head))
```

```
(hyd2
  head
  (hyd2
    (hyd1
      (hyd2 head head))
    head))
```

```
(hyd2
  head
  (hyd2
    (hyd1
      (hyd2 head     ))
    head))
```

```
(hyd2
  head
  (hyd2
    (hyd5
      (hyd1 head)
      (hyd1 head)
      (hyd1 head)
      (hyd1 head)
      (hyd1 head))
    head))
```

```
(hyd2
  head
  (hyd2
    (hyd1
      (hyd2 head head))
    head))
```

```
(hyd2
  head
  (hyd2
    (hyd5
      (hyd1 head)
      (hyd1 head)
      (hyd1 head)
      (hyd1 head)
      (hyd1 head))
    head))
```

```
(hyd2
  head
  (hyd2
      (hyd1 head)
      (hyd1 head)
      (hyd1 head))
    head))
```

Continuity

```
(hyd2
  head
  (hyd2
    (hyd1
      (hyd2 head head))
    head))
```

```
(hyd2
  head
  (hyd2
    (hyd1
      (hyd2 head head))
    head))
```

```
(hyd2
  head
  (hyd2
    head
    (hyd1
      (hyd2 head head))))
```

```
(hyd2
  head
  (hyd2
    (hyd1
      (hyd2 head head))
    head))
```

```
level 0: #0
level 1: 0 -> #1, 0 -> #2
level 2: 2 -> #3, 2 -> #4
level 3: 3 -> #5
level 4: 5 -> #6, 5 -> #7
```

```
(hyd2
  head
  (hyd2
    (hyd1
      (hyd2 head head))
    head))
```

```
level 0: #0
level 1: 0 -> #1, 0 -> #2
level 2: 2 -> #3, 2 -> #4
level 3: 3 -> #5
level 4: 5 -> #6, 5 -> #7
```

```
(hyd2
  head
  (hyd2
    (hyd5
      (hyd1 head)
      (hyd1 head)
      (hyd1 head)
      (hyd1 head)
      (hyd1 head))
    head))
```

```
level 0: #0
level 1: 0 -> #1, 0 -> #2
level 2: 2 -> #3, 2 -> #4
level 3: 3 -> #5, 3 -> #8,
         3 -> #10, 3 -> #12,
         3 -> #14
level 4: 5 -> #6, 8 -> #9,
         10 -> #11,
         12 -> #13,
         14 -> #15
```

```
(hyd2
  head
  (hyd2
    (hyd5
      (hyd1 head)
      (hyd1 head)
      (hyd1 head)
      (hyd1 head)
      (hyd1 head))
    head))
```

```
(hyd2
  head
  (hyd2
    (hyd5
      (hyd1 h
      (hyd1 h
      (hyd1 head)
      (hyd1 head)
      (hyd1 head))
    head))
```

Canonicity

Ideal diagrammatic notation systems

Learn from the success of **text notations**:

Ideal diagrammatic notation systems

Learn from the success of **text notations**:

- Low cost, high reward
- Structure-driven composition
- Continuity
- Canonicity
- Good-looking idioms

Ideal diagrammatic notation systems

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- Canonicity
- Good-looking idioms

What might this look like for **diagrammatic notations**?

We are *not* trying to…

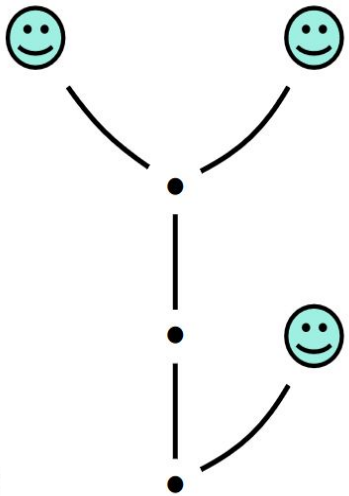
- Describe ‘good’ diagrams
  - Rather, describe good diagramming systems, and illustrate their properties with some concrete diagrams
- Prove our system better than other tools
  - (We don’t have one)
- Prove diagrams better than text notations
  - Rather, computer-assisted proofs should have both!

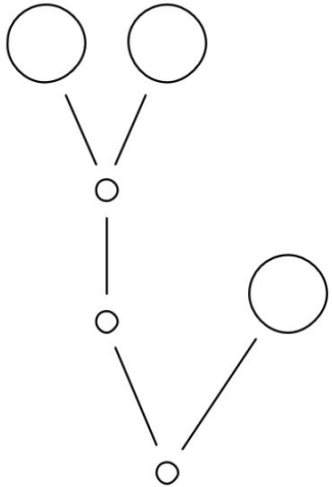
Ideal diagrammatic notation systems

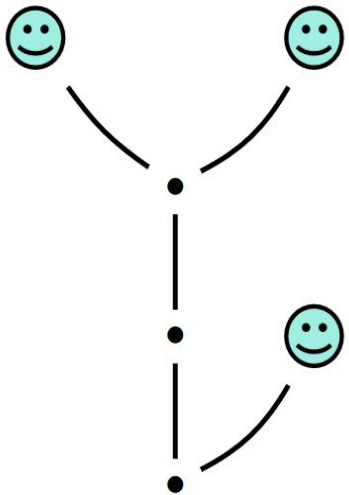
Learn from the success of **text notations**:

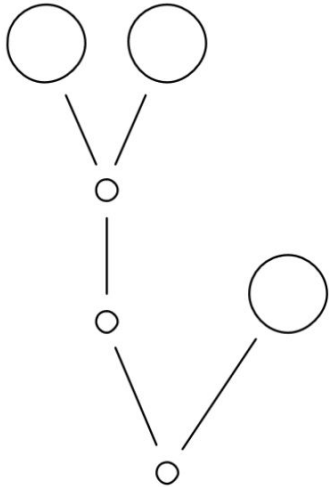
- Low cost, high reward
- Structure-driven composition
- Continuity
- Canonicity
- Good-looking idioms

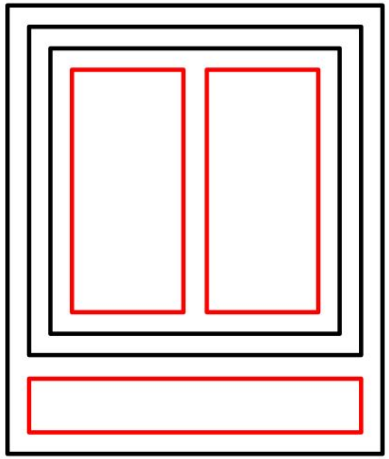
What might this look like for **diagrammatic notations**?











Structure-driven composition

Structure-driven composition

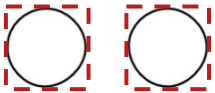
```
(node hnil)
```

Structure-driven composition

```
(node hnil)
```

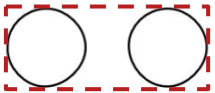
```
diagram (node hnil) := circle
```

Structure-driven composition



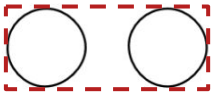
```
(node hnil)
(node hnil)
```

Structure-driven composition



```
(hcons (node hnil)
(hcons (node hnil)
hnil))
```

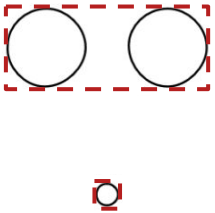
Structure-driven composition



```
(hcons (node hnil)
(hcons (node hnil)
hnil))
```

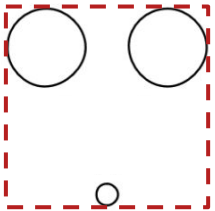
```
diagram (hcons x xs) :=
  happend (diagram x) (diagram xs)
```

Structure-driven composition



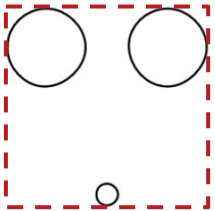
```
node
(hcons (node hnil)
(hcons (node hnil)
hnil))
```

Structure-driven composition



```
      (node
(hcons (node hnil)
(hcons (node hnil)
hnil)))
```

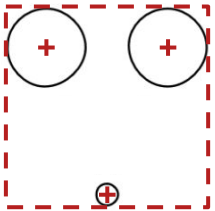
Structure-driven composition



```
      (node
(hcons (node hnil)
(hcons (node hnil)
hnil)))
```

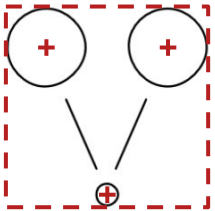
```
diagram (node xs) :=
  vappend circle (diagram xs)
```

Structure-driven composition



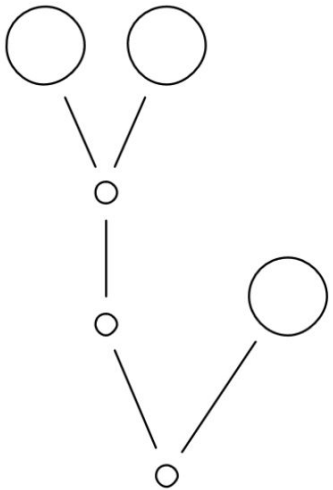
```
(node
(hcons (node hnil)
(hcons (node hnil)
hnil)))
```

Structure-driven composition



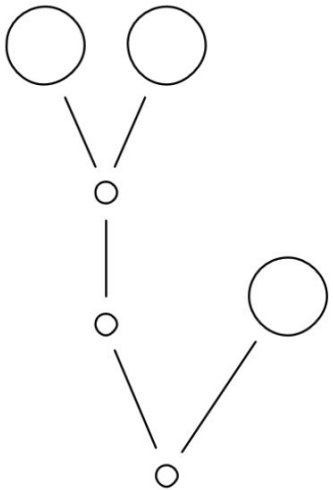
```
      (node
(hcons (node hnil)
(hcons (node hnil)
hnil)))
```

Structure-driven composition



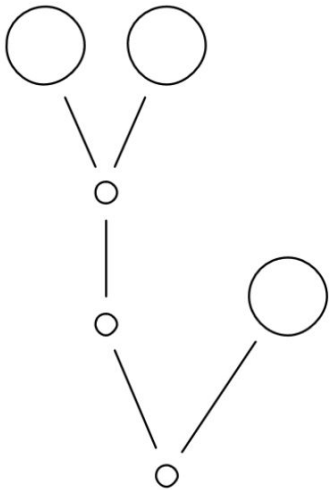
```
diagram (node hnil) := circle

diagram (hcons x xs) :=
  happend (diagram x) (diagram xs)

diagram (node xs) :=
  vappend circle (diagram xs)
```

High reward. Low cost?



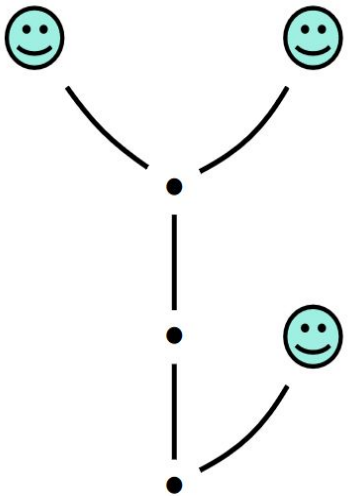
```
diagram (node hnil) := circle

diagram (hcons x xs) :=
  happend (diagram x) (diagram xs)

diagram (node xs) :=
  vappend circle (diagram xs)
```

High reward. Low cost?



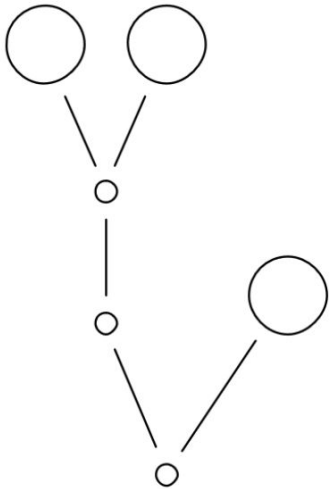
```
diagram (node hnil) := circle

diagram (hcons x xs) :=
  happend (diagram x) (diagram xs)

diagram (node xs) :=
  vappend circle (diagram xs)


Infix "\*" := sep.
```

High reward. Low cost?

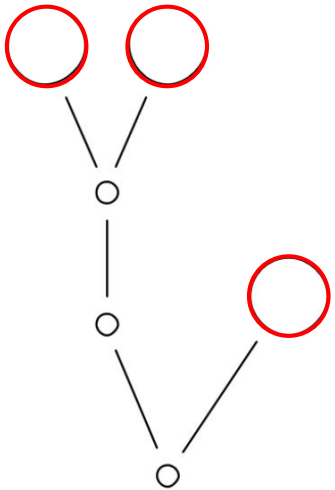


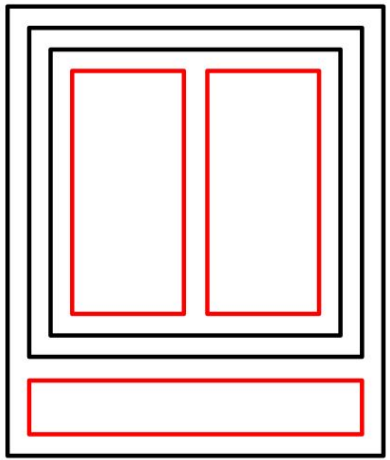
```
\node (N1) at (2,2) {$\bullet$};
\node (N2) at (2,4) {$\bullet$};
\node (N3) at (2,6) {$\bullet$};
\node (H1) at (0,8)
  {$\Smiley[1.5][vertfluo]$};
\node (H2) at (4,8)
  {$\Smiley[1.5][vertfluo]$};
\node (H5) at (4,4)
  {$\Smiley[1.5][vertfluo]$};
\draw (N1) -- (N2);
\draw (N2) -- (N3);
\draw (N3) to [bend left=10]  (H1);
\draw (N3) to [bend right=16] (H2);
\draw (N1) to [bend right=16] (H5);
```

Good-looking idioms

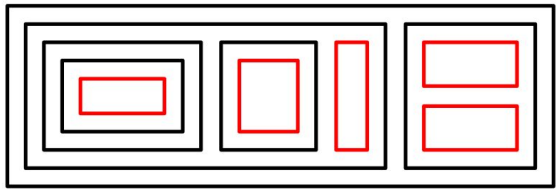


Good-looking idioms

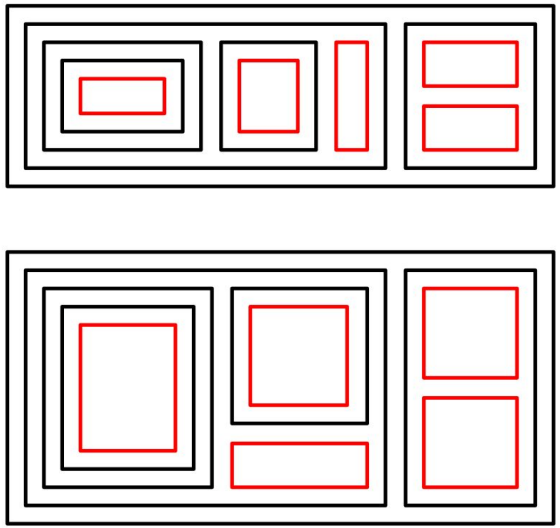




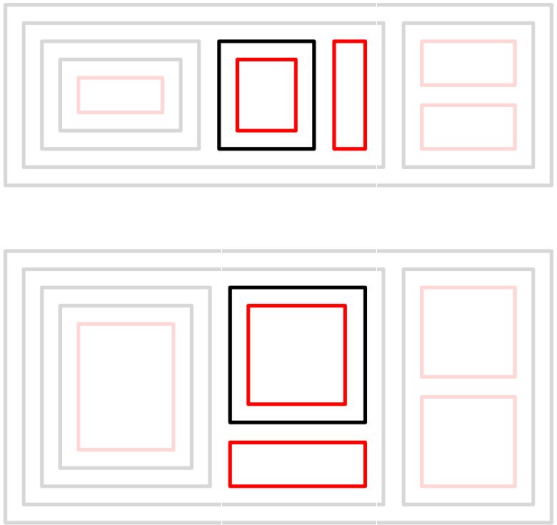
Good-looking idioms



Good-looking idioms

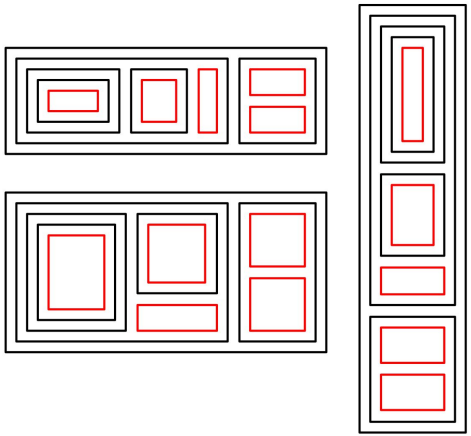


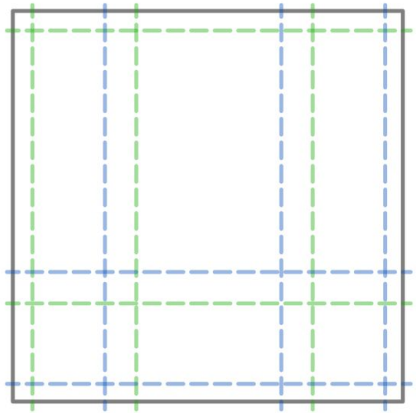
Good-looking idioms



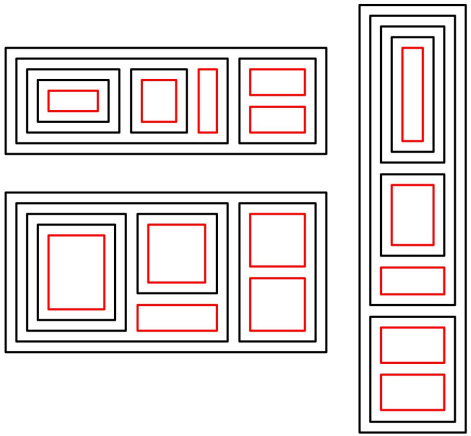
Good-looking idioms

Good-looking idioms

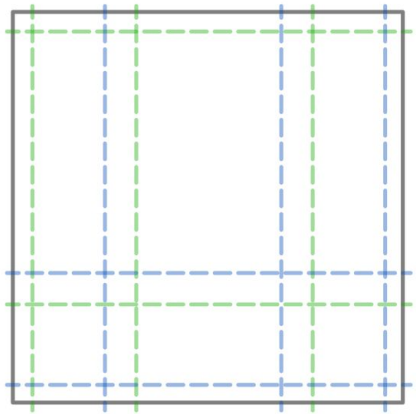




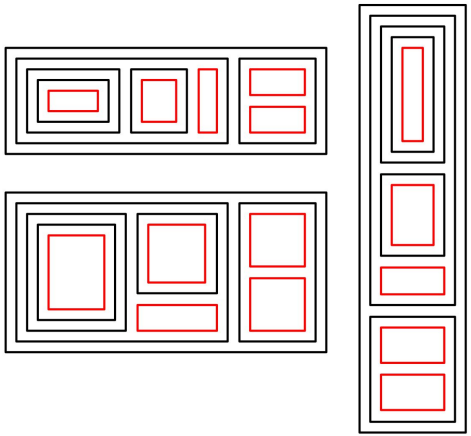
Good-looking idioms



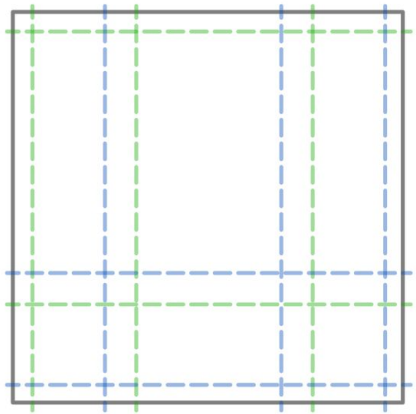
```
(define gl
  (new grid-layout [...]

                                  ))
```



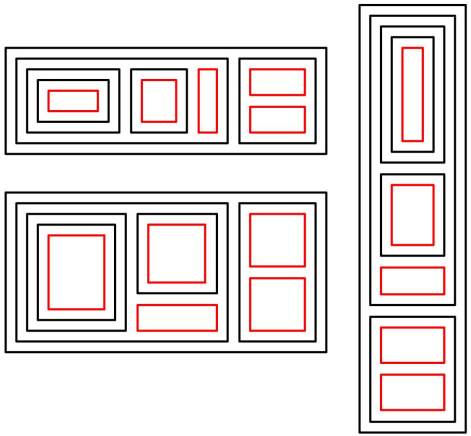
Good-looking idioms



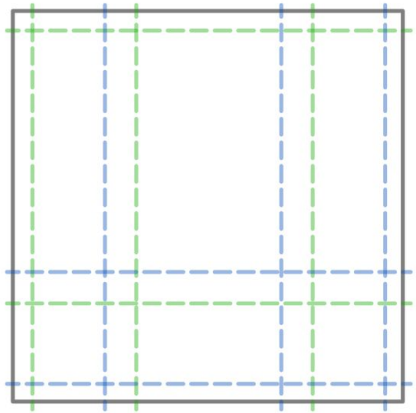
```
(define gl
  (new grid-layout
                ; columns  rows
    [grid-defn '((1 2 1)  (3 1))]
                       ))
```



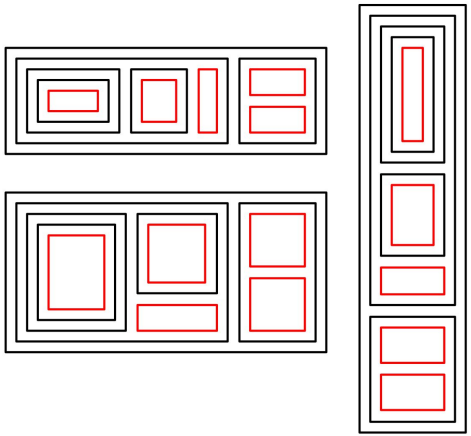
Good-looking idioms



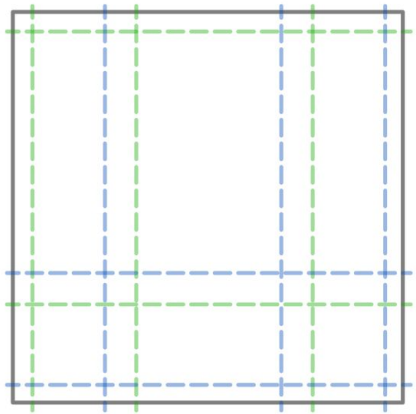
```
(define gl
  (new grid-layout
                ; columns  rows
    [grid-defn '((1 2 1)  (3 1))]
    [gap 16] [padding 10]))
```



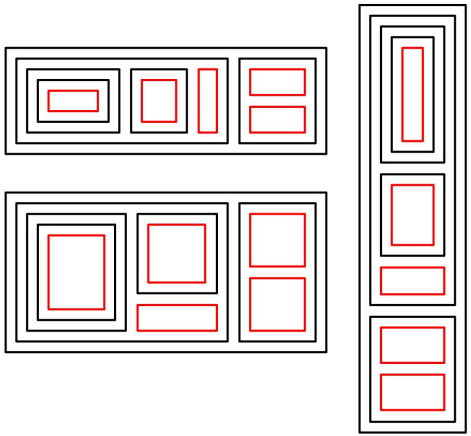
Good-looking idioms



```
(define gl
  (new grid-layout
                ; columns  rows
    [grid-defn '((1 2 1)  (3 1))]
    [gap 16] [padding 10]))


(send gl draw! 200 200)
```

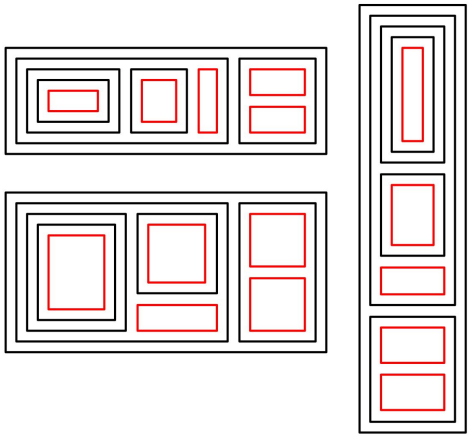


Good-looking idioms

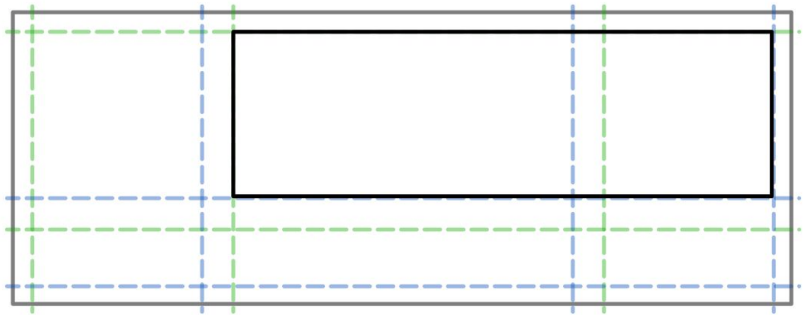


```
(define gl
  (new grid-layout
                ; columns  rows
    [grid-defn '((1 2 1)  (3 1))]
    [gap 16] [padding 10]))


(send gl draw! 400 150)
```

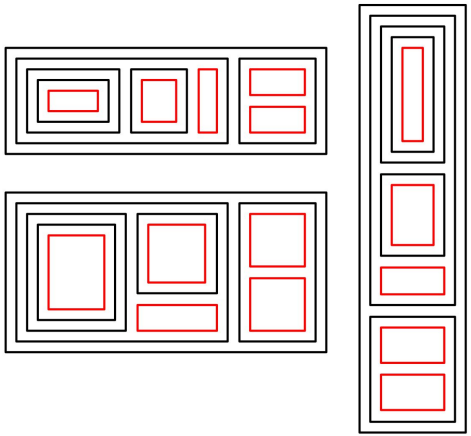
Good-looking idioms



```
(define gl
  (new grid-layout
                ; columns  rows
    [grid-defn '((1 2 1)  (3 1))]
    [gap 16] [padding 10]
    [item-defns `((0 1 1 2 ,box))]))

(send gl draw! 400 150)
```

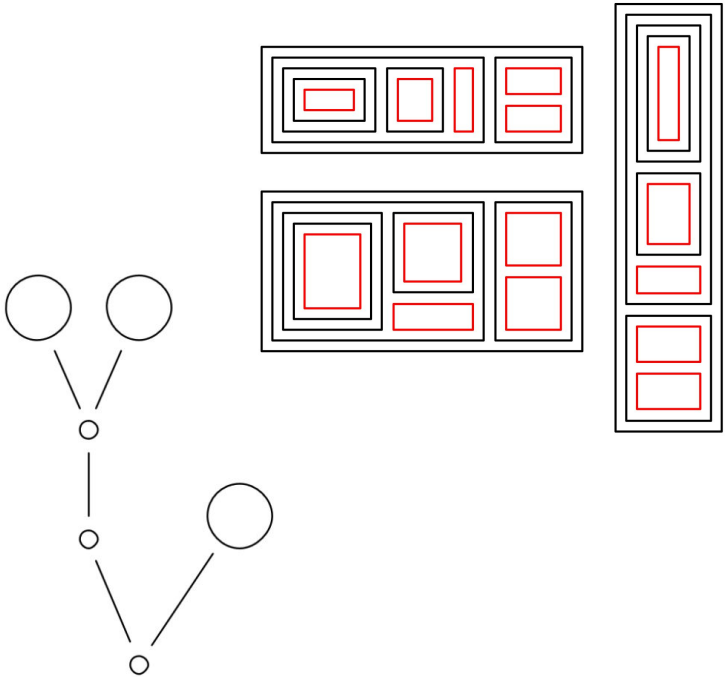


Good-looking idioms

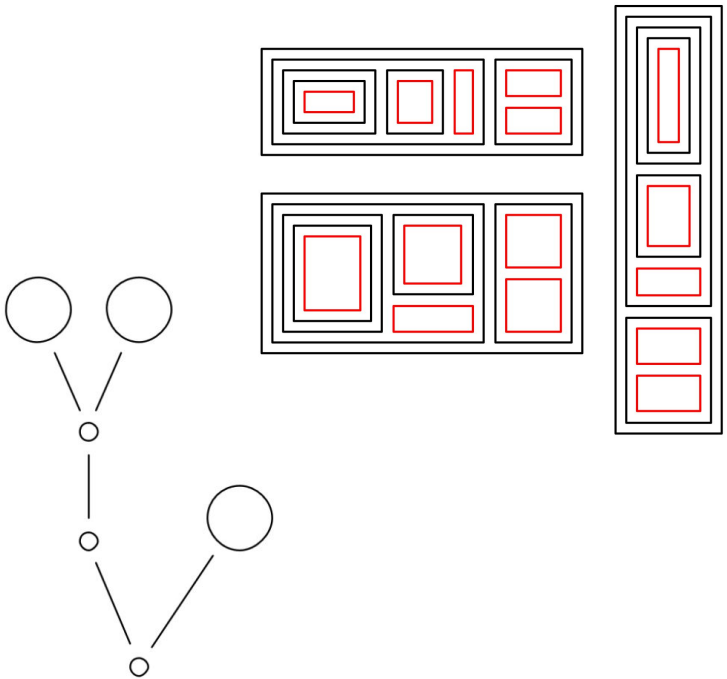


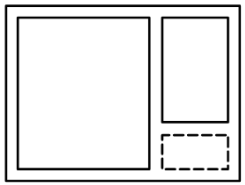
```
(define gl
  (new grid-layout
                ; columns  rows
    [grid-defn '((1 2 1)  (3 1))]
    [gap 16] [padding 10]
    [item-defns `((0 1 1 2 ,box))]))

(send (send gl transpose) draw! 400 150)
```

Good-looking idioms

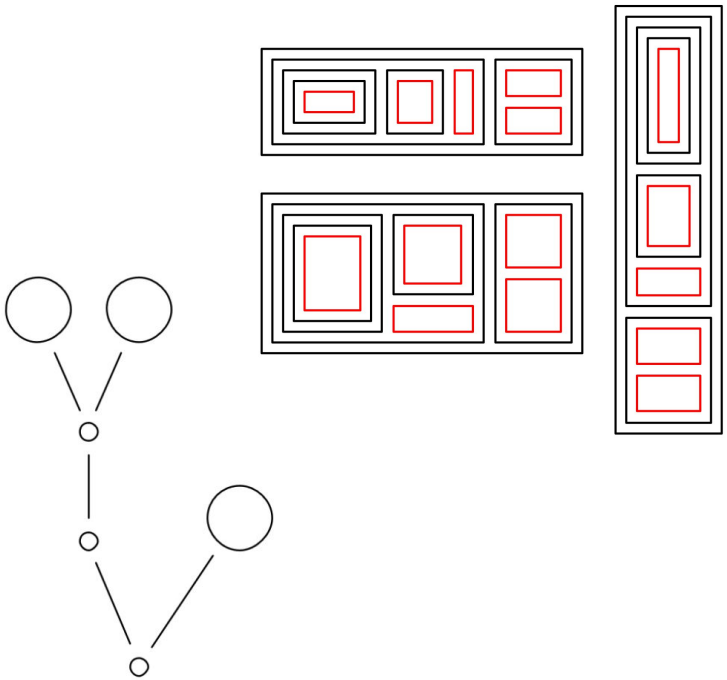


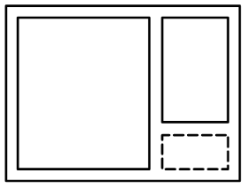
Good-looking idioms



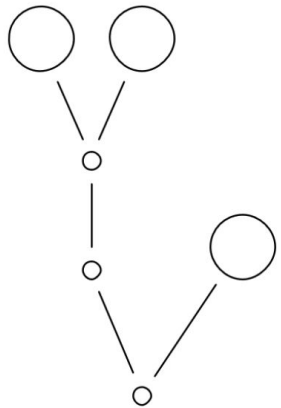
- Grid layouts
  - Transpositions
  - 

Good-looking idioms

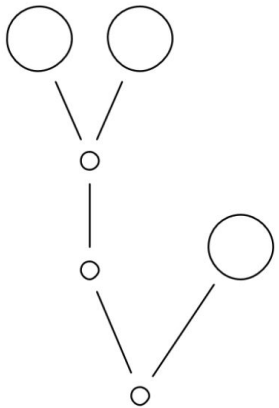


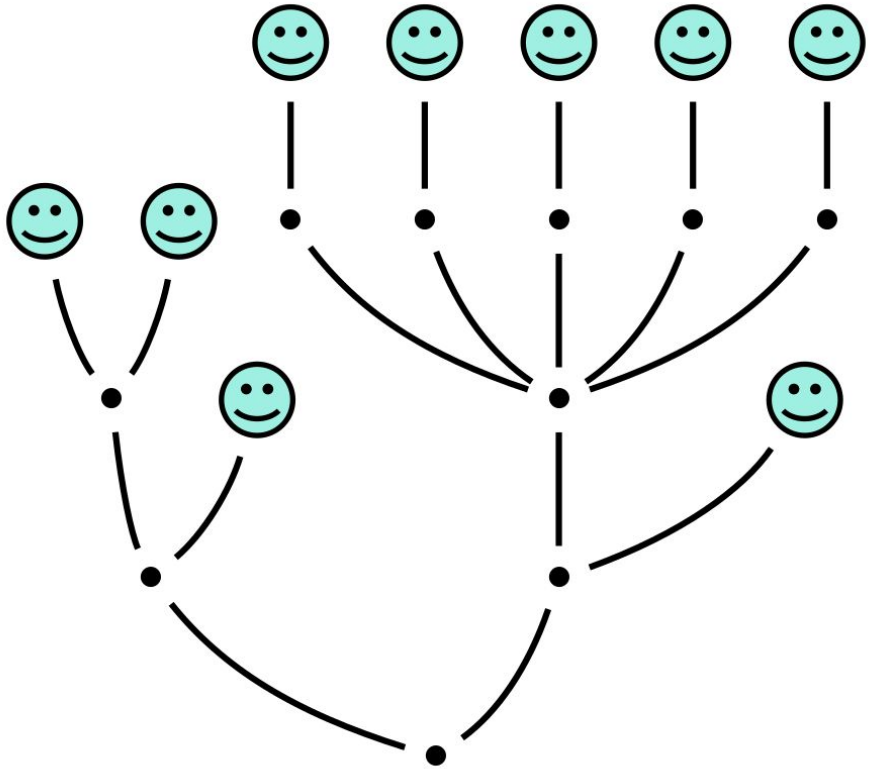
- Grid layouts
  - Transpositions
  - 
- Diagram composition
  - horizontal & vertical juxtaposition
  - bounding boxes
  - endpoints & connectors
  - nesting

Good-looking idioms

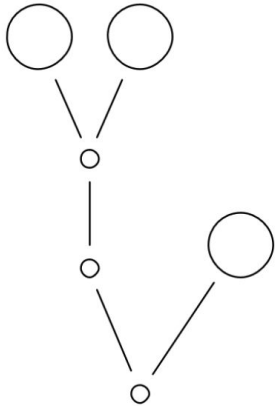


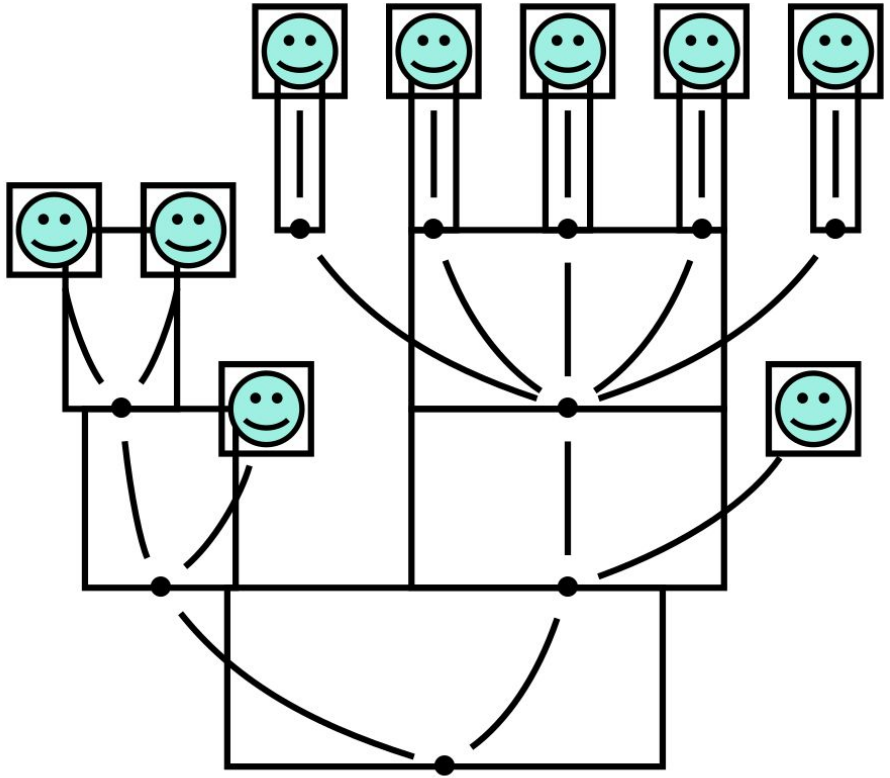
Good-looking idioms



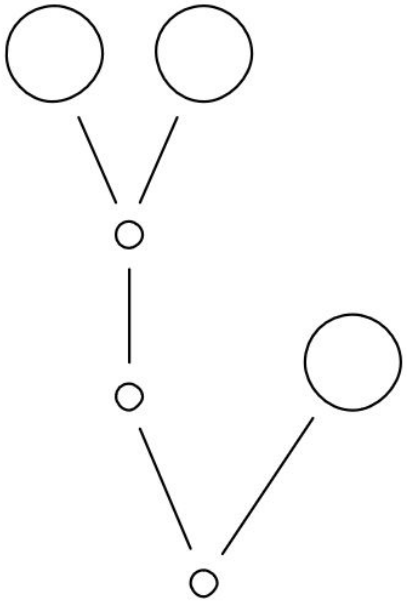


Good-looking idioms

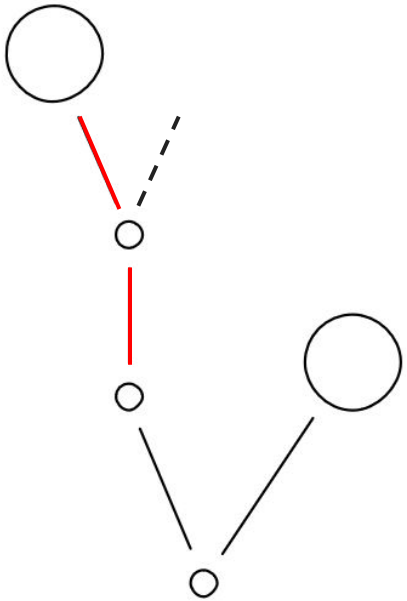




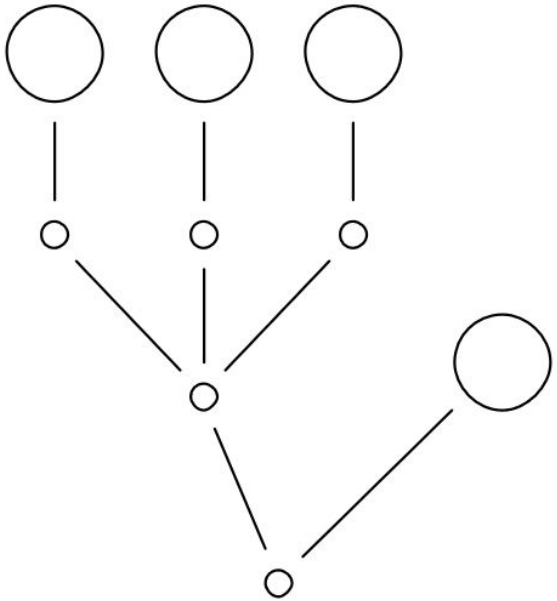
Continuity



Continuity



Continuity



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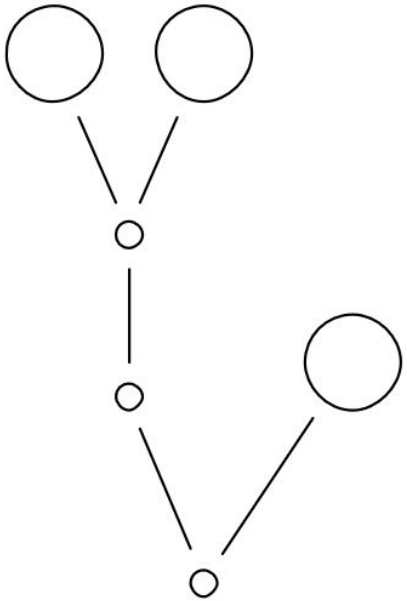
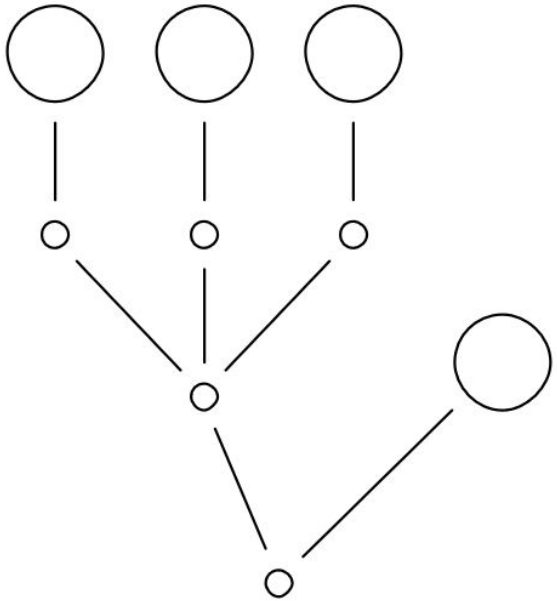
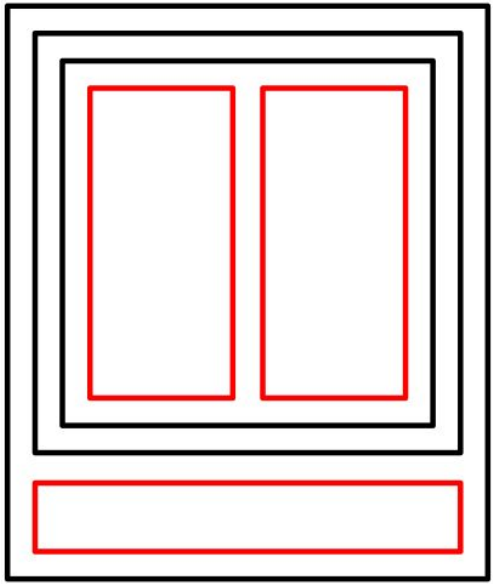


Continuity



Continuity



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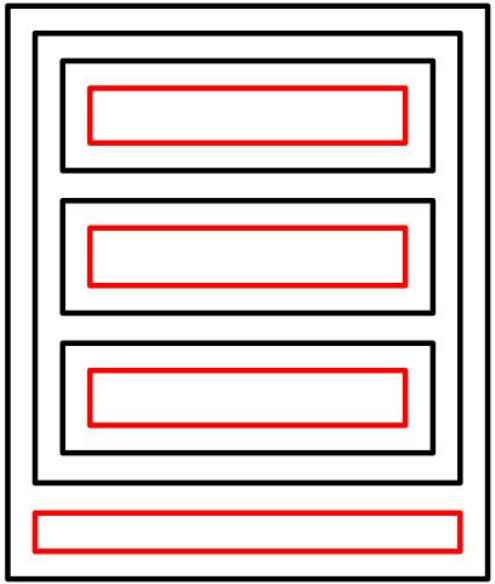
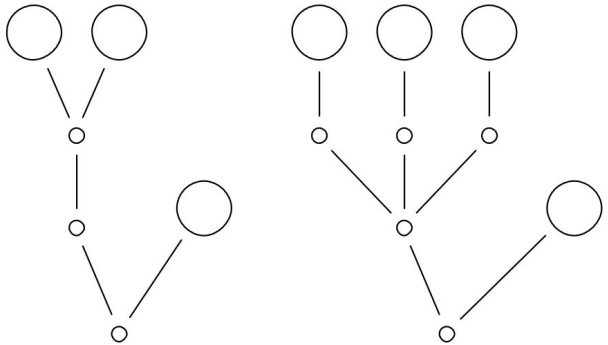


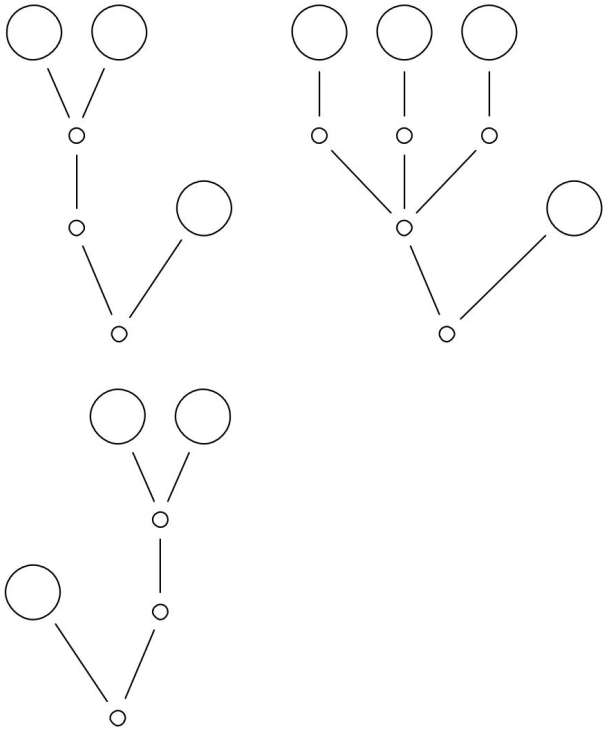
Canonicity

Canonicity



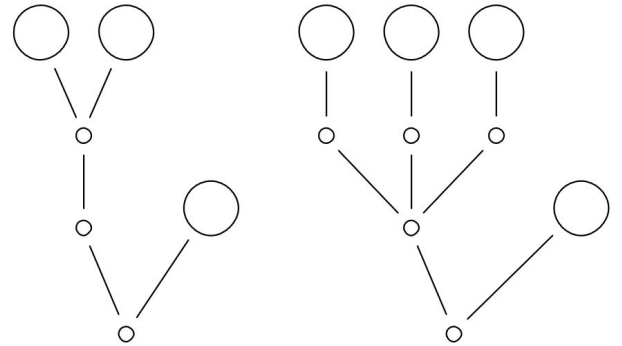
- One possible diagram per hydra
- Children are ordered, visually and syntactically

Canonicity

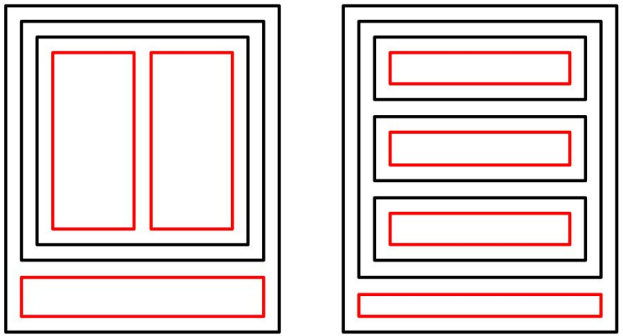


- One possible diagram per hydra
- Children are ordered, visually and syntactically

Canonicity



- One possible diagram per hydra
- Children are ordered, visually and syntactically



- Many possible diagrams per hydra
- Syntactic order ↔ visual order?

Conclusion

- We need diagrammatic notations in computer-assisted proofs, but they are mostly missing.

Conclusion

- We need diagrammatic notations in computer-assisted proofs, but they are mostly missing. Why?

Conclusion

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- Let’s learn from the success of **text notations**.

Conclusion

- We need diagrammatic notations in computer-assisted proofs, but they are mostly missing. Why?
- Let's learn from the success of **text notations**.
  - Low cost, high reward
  - Structure-driven composition
  - Continuity
  - Canonicity
  - Good-looking idioms

Conclusion

- We need diagrammatic notations in computer-assisted proofs, but they are mostly missing. Why?
- Let's learn from the success of **text notations**.
  - Low cost, high reward
  - Structure-driven composition
  - Continuity
  - Canonicity
  - Good-looking idioms
- An ideal **diagrammatic notation** system translates these properties to diagrams.

Our plan

- Study diagram language design.
  - for graphs (math & stats)
  - for individual diagrams vs. notations
- Study diagram workflows.
  - interactivity
  - theorem prover integration
- Study diagrammatic calculi.
- Build stuff.

Thank you!

Questions?

Diagrammatic notations for ITP

- We need diagrammatic notations in computer-assisted proofs, but they are mostly missing. Why?
- Let's learn from the success of **text notations**.
  - Low cost, high reward
  - Structure-driven composition
  - Continuity
  - Canonicity
  - Good-looking idioms
- An ideal **diagrammatic notation** system translates these properties to diagrams.

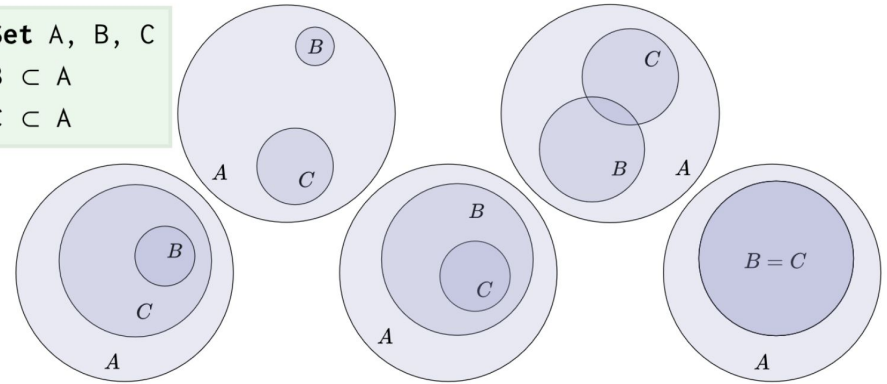
Penrose

- Moderate cost, high reward:
  Offline generation of **individual** diagrams
- Structure-driven composition:
  “Domain” + “stylesheet” = “substance” → diagram
- Continuity: non-goal
- Canonicity: non-goal
- Good-looking idioms

Penrose

```
forall Set x; Set y
where IsSubset(x, y) {
    ensure contains(y.shape, x.shape)
    ensure smallerThan(x.shape, y.shape)
    ensure outsideOf(y.text, x.shape)
    layer x.shape above y.shape
    layer y.text below x.shape
}
```

```
Set A, B, C
B ⊂ A
C ⊂ A
```



ProofWidgets (Lean 4)

- High cost, high reward:
  Fully general HTML+JS integration
- Structure-driven composition, continuity, canonicity:
  Largely up to the UI programmer
- Good-looking idioms:
  UI components, interactivity, extensibility

ProofWidgets (Lean 4)

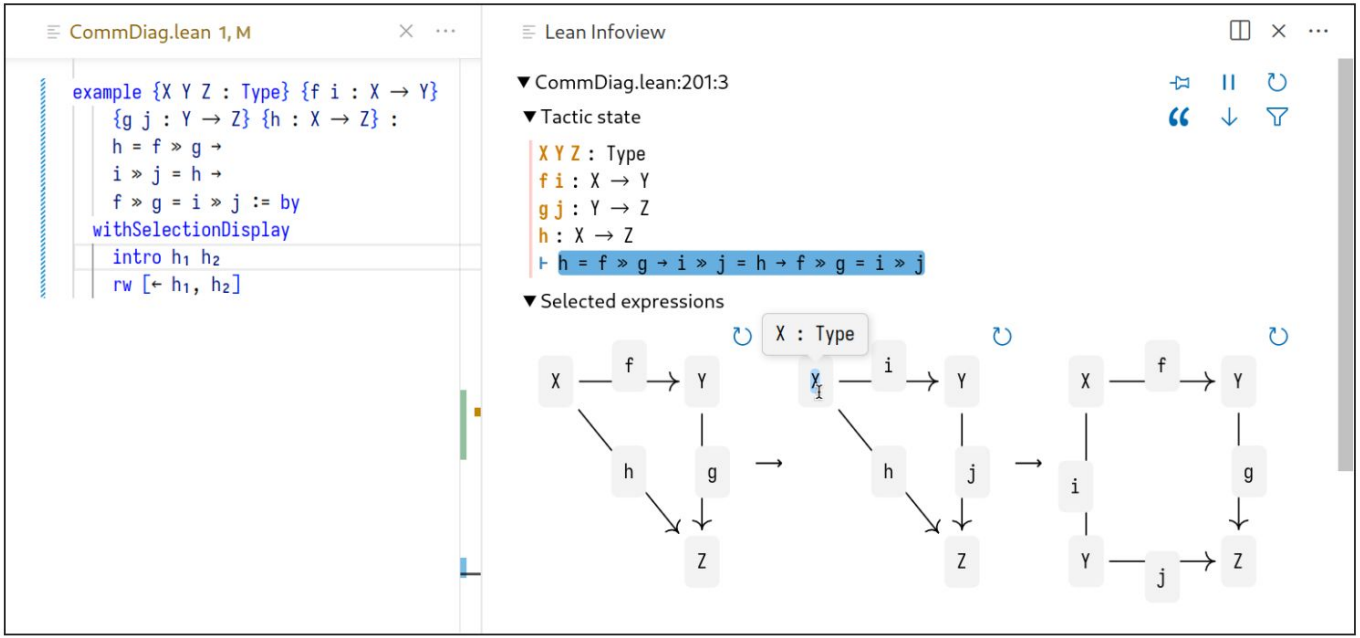


Figure 4 A target type in the language of category theory is selected. The statement is displayed as a sequence of commutative diagrams connected by implication arrows.

D3.js

- High cost, high reward:
  “D3 is for bespoke visualization. D3 makes things possible, not necessarily easy; […] D3 makes sense for media organizations […] where a single graphic may be seen by a million readers”
- Structure-driven composition:
  “Data-driven diagrams”
- Continuity, canonicity: ??
- Good-looking idioms:
  Are the soul of D3

Hydras & Co.

- Textbook written in literate Coq style with Alectryon

Likewise, the *height* (maximum distance between the foot and a head) is defined by mutual recursion:

```
Fixpoint height  (h:Hydra) : nat :=
  match h with
  | node l => lheight l
  end
with lheight l : nat :=
       match l with
       | hnil => 0
       | hcons h hs => Nat.max (S (height h)) (lheight hs)
       end.

  Compute height Hy.
```

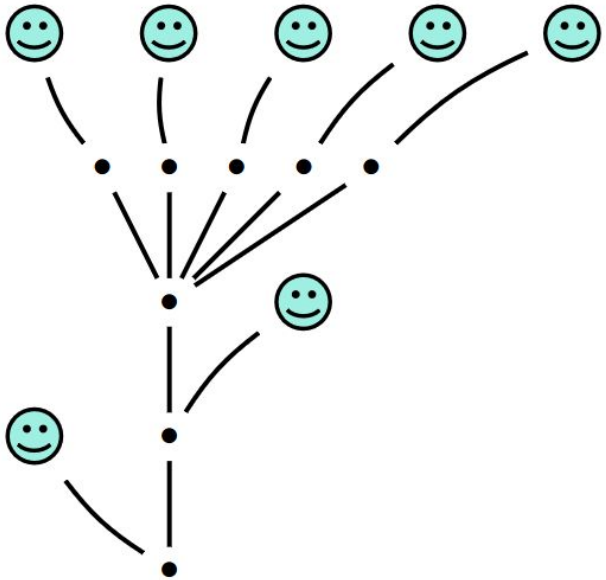
```
= 4
: nat
```

Hydras & Co.

Let us consider for instance the hydra `Hy''` of Fig 2.5 on page 24.

From Module Hydra.Hydra_Examples

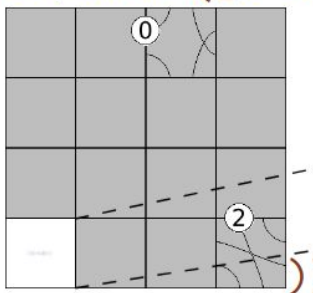
```
Example Hy'' :=
  hyd2 head
       (hyd2
          (hyd_mult (hyd1 head) 5)
          head).
```



Interactive visual syntax (Andersen et al., 2020)

- Turo game tiles as part of Racket syntax

```
; Tile -> [Listof Board]
(define (all-possible-configurations t)
  (for/list ([d DEGREES])
```



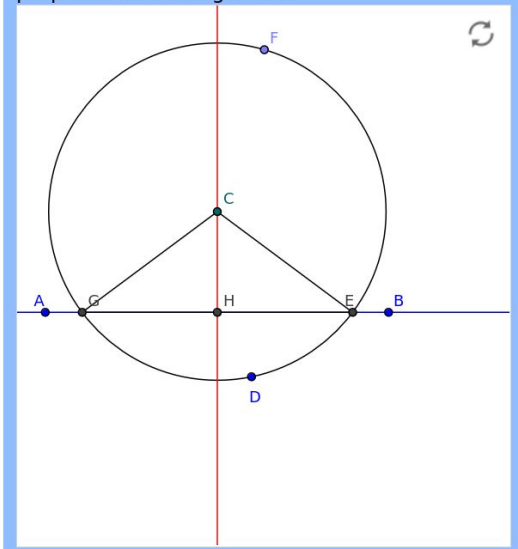

```
))
```

```
(send t rotate d)
```

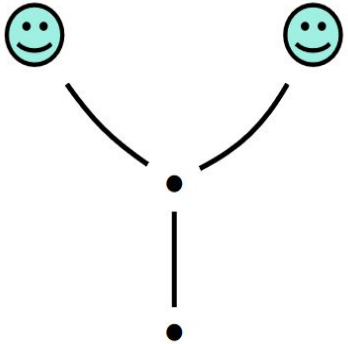
GeoCoq

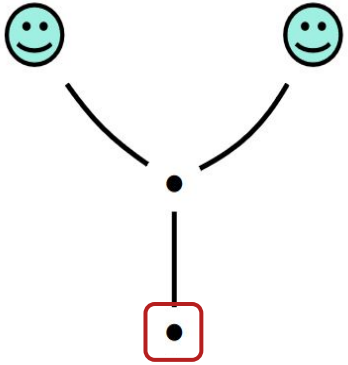
Proposition 12

To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line.



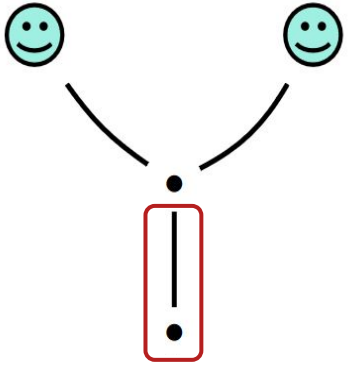

```
Lemma prop_12 : ∀ A B C, ¬ Col A B C →
  ∃ X : Tpoint, Col A B X ∧ Perp A B C X.
Proof.
  apply l8_18_existence.
Qed.
```



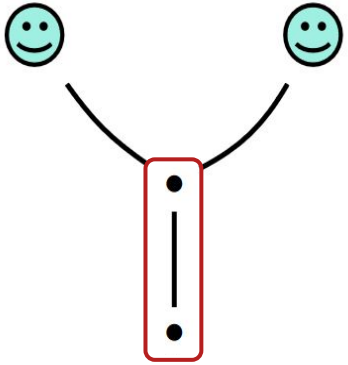


(node

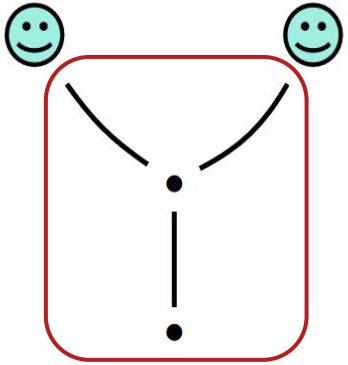
)



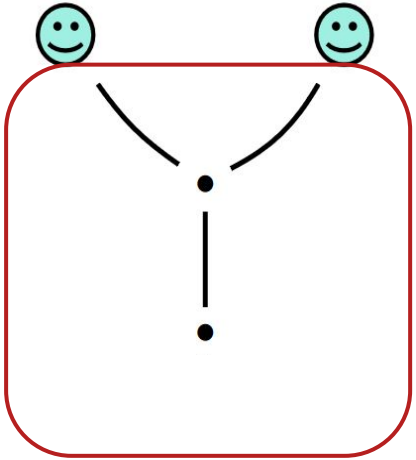
```
(node
  (hcons

   hnil))
```



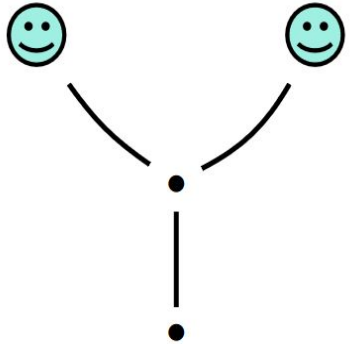
```
(node
  (hcons (node
            )
   hnil))
```



```
(node
  (hcons (node
     (hcons
     (hcons
      hnil)))
   hnil))
```

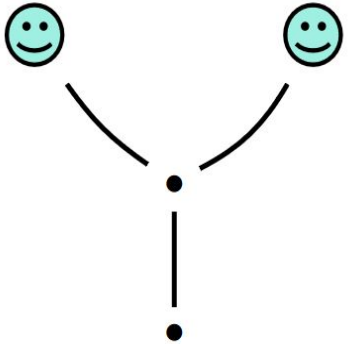


```
(node
  (hcons (node
     (hcons (node hnil)
     (hcons (node hnil)
      hnil)))
   hnil))
```



```
(node
  (hcons (node
     (hcons (node hnil)
     (hcons (node hnil)
      hnil)))
   hnil))
```

```
Notation head := (node hnil).
Notation hyd1 h := (node (hcons h hnil)).
Notation hyd2 h h' := (node (hcons h (hcons h' hnil))).
Notation hyd3 h h' h'' := (node (hcons h (hcons h' (hcons h'' hnil)))).
```



```
(hyd1 (hyd2 head head))
```

```
Notation head := (node hnil).
Notation hyd1 h := (node (hcons h hnil)).
Notation hyd2 h h' := (node (hcons h (hcons h' hnil))).
Notation hyd3 h h' h'' := (node (hcons h (hcons h' (hcons h'' hnil)))).
```